Fiber products

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One of the most fundamental properties of schemes is the unrestricted existent of fiber products. The fiber product is extremely useful in many situations and takes on astonishingly versatile roles. We begin the paragraph with recalling the definition of the fibre product of sets, then slide into a very general situation to discuss fibre product in general categories, for then to return to the present context of schemes. We prove the existence theorem, and finish up by discussing a series of examples.

Fiber products of sets. As a warming up we use some lines on recalling the fiber product in the category **Sets** of sets. The points of departure is two sets X_1 and X_2 both equipped with a map to a third set S; *i.e.,* we are given a diagram

$$\begin{tikzcd} X_1 \arrow[dr, "\psi_1"'] & & X_2 \arrow[dl, "\psi_2"] \\ & S & \end{tikzcd}$$

The fibre product $X_1 \times_S X_2$ is the subset of the cartesian product $X \times Y$ consisting of the pairs whose two components have the same image in S; that is, we have

$$X_1 \times_S X_2 = \{\, (x_1, x_1) \mid \psi_1(x_1) = \psi_2(x_2) \,\}.$$

Clearly the diagram below where π_1 and π_2 denote the restrictions of two projections to the fiber product—that is, $\pi_i(x_1, x_2) = x_i$— is commutative,

$$\begin{tikzcd} & X_1 \times_S X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 \arrow[dr, "\psi_1"'] & & X_2 \arrow[dl, "\psi_2"] \\ & S & \end{tikzcd} \tag{1}$$

And more is true; the fibre product enjoys a universal property: Given any two maps $\phi_1 \colon Z \to X_1$ and $\phi_2 \colon Z \to X_2$ such that $\psi_1 \circ \phi_{X_1} = \psi_2 \circ \phi_2$ there is a unique map $\phi \colon Z \to X_1 \times_S X_2$ such that $\pi_1 \circ \phi = \phi_1$ and $\pi_2 \circ \phi = \phi_2$. To lay your hans on such an ϕ, just use the map whose two components are ϕ_1 and ϕ_2 and observe that it takes values in $X_1 \times_S X_2$ since the relation $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ holds. The data of the two ϕ_i's is to give a commutative diagram like 1 above with Z replacing the product $X \times_S Y$, and the universal property is to say that universal—a more precise usage would be to say it is *final* (final)—among such diagrams.

The name *fiber product* stems from t the fiber of the map $\psi = \psi_i \circ \pi_i$ from $X \times_S Y$ to S over a point $s \in S$ just being the direct product of the fibers of ψ_1 and ψ_2, that is $\psi^{-1}(s) = \psi_1^{-1}(s) \times \psi_2^{-1}(s)$.

THE FIBER PRODUCT IN GENERAL CATEGORIES The notion of a fiber product—formulated as the solution to a universal problem as above—is *mutatis mutandis* meaningful any category C. Given any two arrows $\psi_i\colon X_i \to S$ in the category C. An object—that we shall denote by $X_1 \times_S X_2$—is said to be the *fiber product* (fiberproduktet) of the objects X_i—or more precisely of the two arrows $\psi_i\colon X_i \to S$—if the following two conditions are fulfilled:

- There are two arrows $\pi_i\colon X_1 \times_S X_2 \to X_i$ in C such that $\psi_1 \circ \pi_1 = \psi_2 \circ \pi_2$ (called the projections).
- For any two arrows $\phi_i\colon X \to X_i$ in C such that $\phi \circ \psi_1 = \phi \circ \psi_2$, there is a unique arrow $\phi\colon X \to X_1 \times_S X_2$ such that $\pi_i \circ \phi = \phi_i$ for $i = 1, 2$.

If the fiber product exists, it is unique up to a unique isomorphism as is true for a solution to any universal problem. However, it is a good exercise to check this in detail in this specific situation.

PROBLEM 0.1. Show that if the fiber product exists in the category C, it is unique up to a unique isomorphism. ✱

To any object X in C recall that one has the covariant functor $h_X\colon \mathsf{C} \to \mathsf{Sets}$ that to any object T in C associates the set $h_X(T) = \mathrm{Hom}_C(T, X)$ and to any arrow $\alpha\colon T' \to T$ in C associates the map $h_X(\alpha)\colon h_X(T) \to h_X(T')$ sending f to $f \circ \alpha$. The given arrows ψ_i gives rise to maps of functors[1] $h_{\psi_i}\colon h_{X_i} \to h_S$ sending a arrow $f \in h_{X_i}(T)$ to the composition $\psi_i \circ f$. The universal property of the fiber product translates into the following. For any object T in C, one has an equivalence of functors from C (or isomorphism if you want):

$$h_{X_1 \times_S X_2} \xrightarrow{\simeq} h_{X_1} \times_S h_{X_2} \tag{2}$$

where the arrow sends an arrow $\psi \in h_{X_1 \times_S X_2}(T)$ to the pair of arrows $\pi_i \circ \psi$. The content in this formulation says is that for any object T in the category C, the set $\mathrm{Hom}_C(T, X_1 \times_S X_2)$ of arrows into the fiber product is the fiber product of the two sets $\mathrm{Hom}_C(T, X_1)$ and $\mathrm{Hom}_C(T, X_{12})$over $\mathrm{Hom}_C(T, S)$, an observation that sometimes is useful.

Sign seen in the vicinity of certain German universities

[1]Normally these are called natural transformstiond.

Some general notation The two arrows $\pi_X \circ \phi$ and $\pi_2 \circ \phi$ that determine the arrow $\phi\colon T \to X \times_S Y$ are called *the components* (komponentene) of ϕ, and the notation $\phi = (\phi_1, \phi_2)$ is current. If $\phi_1\colon Y_1 \to X_1$ and $\phi_2\colon Y_2 \to X_2$ are two arrows over S, there is a unique arrow denoted $\phi_1 \times \phi_2$ from $Y_1 \times_S Y_2$ to $X_1 \times_S X_2$ whose components are $\phi_1 \circ \pi_{Y_1}$ and $\phi_2 \circ \pi_{Y_2}$.

There is a slightly different approach to the fiber product based on the category of so called *objects over S* where S an object in C. The objects in this new category are arrows $\psi\colon X \to S$ from C and new arrows from $\psi_X\colon X \to S$ to $\psi_Y\colon Y \to S$ are arrows $\phi\colon X \to Y$ rendering the following diagram commutative

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\psi_X"'] & & Y \arrow[dl, "\psi_Y"] \\ & S & \end{tikzcd}$$

This new category is denoted C/S. If $\psi_X\colon X \to S$ is an object from C/S one uses the shorthand notation X/S for $\psi\colon X \to S$; the map ψ_X being understood. One furthermore puts $h_{X/S}(T/S) = \operatorname{Hom}_{\mathsf{C}/S}(T/S, X/S)$, and with these conventions the relation (2) takes the form

$$h_{X \times_S Y/S} = h_{X/S} \times h_{Y/S}.$$

If $T \to S$ is an arrow and X/S is an object in C/S the fiber product $X_T = X \times_S T$ is an object in C/T with the projection onto t as the structural arrow. In case fiber products over S exists unrestrictedly in C, this give a functor $X \to X_T$ from C/S to C/T; it acts arrow by sending the arrow $\phi\colon Y \to X$ in C/S to the arrow $\phi_T = \phi \times \mathrm{id}_T$ that goes from $Y_T \to X_T$. The object X_T and the arrow ϕ_T are often called the *pull backs* (tilbaketrekningene) of X respectively ϕ. Another frequently used terminology is to say that X_T is obtained from S by *the base change* (basisforandringen) $T \to S$, which is quite a natural notion when thinking of S as the base object in C/S.

Any arrow $\phi\colon X \to Y$ induces a natural transformation (or map of functors) $\phi^*\colon h_X \to h_Y$ just by composition; *i.e.*, ϕ^* sends an arrow $\alpha\colon T \to X$ to $\phi \circ \alpha$. One easily verifies that $(\phi_1 \circ \phi_2)^* = \phi_1^* \circ \phi_2^*$ so the association is functorial.

A simple, but from time to time very useful result is the so called *Yoneda lemma*. It says that the association $\phi \to \phi^*$ is a bijection between the set of arrows $X \to Y$ and the set of natural transformations $h_X \to h_Y$. The nice thing is that you can work with functors of type h_X, whose values are good old sets, and if achieve constructing a map between them, you have got an arrow in C.

Problem 0.2. Prove the Yoneda lemma. ✷

Problem 0.3. Let $\phi\colon Y \to X$ be an arrow in C. One says that ϕ is *injective* or a *monomorphism* if $\phi \circ \alpha_1 = \phi \circ \alpha_2$ entails $\alpha_1 = \alpha_2$ for any pair of arrows $\alpha_i\colon T \to Y$. Show that ϕ is a monomorphism if and only if ϕ^* is injective. ✷

PROBLEM 0.4. Assume that $\phi_i \colon Y_i \to X$ for $i = 1, 2$ are two monomorphism. Assume that $h_{Y_1}(T) \subseteq h_{Y_2}(T)$ for all T. Show that one may factor $\phi_2 = \phi \circ \phi_1$ for a unique monomorphism $\phi \colon X_1 \to X_2$. ✱

Products of affine schemes

The category Aff of affine schemes, is more or less by definition, equivalent to the category of rings, and in the category of rings we have the tensor product. The tensor product enjoys a universal property *dual* to the one of fibered product. To be precise, assume that A_1 and A_2 are B-algebras, *i.e.,* we have two maps of rings α_i

$$\begin{tikzcd} A_1 & & A_2 . \\ & B \arrow[ul, "\alpha_2"] \arrow[ur, "\alpha_1"'] & \end{tikzcd}$$

There are two maps $\beta_i \colon A_i \to A_1 \otimes_B A_2$ sending $a_1 \in A_1$ to $a_1 \otimes 1$, respectively sending a_2 to $1 \otimes a_2$. Both are ring homomorphism since $aa' \otimes 1 = (a \otimes 1)(a' \otimes 1)$ respectively $1 \otimes aa' = (1 \otimes a)(1 \otimes a')$, and they fit into the following commutative diagram as $\alpha_1(b) \otimes 1 = 1 \otimes \alpha_2(b)$ by the definition of the tensor product $A_1 \otimes_B A_2$ (this is the significance of the tensor product being taken over B; one can move elements in B from one side of the $\otimes$-glyph to the other).

$$\begin{tikzcd} & A_1 \otimes_B A_2 & \\ A_1 \arrow[ur, "\beta_1"] & & A_2 \arrow[ul, "\beta_2"'] \\ & B \arrow[ul, "\alpha_2"] \arrow[ur, "\alpha_1"'] & \end{tikzcd} \tag{3}$$

Moreover, the tensor product is *universal* in this respect. Indeed, assume that $\gamma_i \colon A_i \to C$ are B-algebra homomorphisms, *i.e.,* $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$; or said differently, they fit into the commutative diagram analogous to (3) with the β_i's replaced by the γ_i's. The association $a_1 \otimes a_2 \to \gamma_1(a_1)\gamma(a_2)$ is bi-B-linear, and hence it extends to a B-algebra homomorphism $\gamma \colon A_1 \otimes_B A_2 \to C$, that obviously have the property that $\gamma \circ \beta_i = \gamma_i$.

Applying the Spec-functor to all this, we get the diagram

$$\begin{tikzcd} & \operatorname{Spec}(A_1 \otimes_B A_2) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \operatorname{Spec} A_1 \arrow[dr] & & \operatorname{Spec} A_2 \arrow[dl] \\ & \operatorname{Spec} B & \end{tikzcd} \tag{4}$$

and the affine scheme $\operatorname{Spec}(A_1 \otimes_B A_2)$ enjoys the property of being universal among affine schemes sitting in a diagram like 4. Hence $\operatorname{Spec}(A_1 \otimes_B A_2)$ equipped with the two projections π_1 and π_2 is the fibered product in the category Aff of affine schemes. One even has the stronger statement that is the fiber product in the bigger category Sch of schemes.

Proposition 0.1 *Given $\phi_i\colon \operatorname{Spec} A_i \to \operatorname{Spec} B$. Then $\operatorname{Spec}(A_1 \otimes_B A_2)$ with the two projection π_1 and π_2 defined as above is the fiber product of the $\operatorname{Spec} A_i$'s in the category of schemes. That is, if Z is a scheme and $\psi_i\colon Z \to \operatorname{Spec} A_i$ are morphisms with $\psi_1 \circ \pi_1 = \psi_2 \circ \phi_2$, there exists a unique morphism $\psi\colon Z \to \operatorname{Spec}(A_1 \otimes_B A_2)$ such that $\pi_i \circ \psi = \psi_i$ for $i = 1, 2$.*

Proof: We know that the proposition is true whenever Z is an affine scheme; so the salient point is that Z not necessarily is affine. For short, we let $X = \operatorname{Spec}(A_1 \otimes_B A_2)$. The proof is just an application of the glueing lemma for morphisms. One covers Z by open affine U_α and covers intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by open affine subsets $U_{\alpha\beta\gamma}$ as well. By the affine case of the proposition, for each U_α we get a map $\psi_\alpha\colon U_\alpha \to X$, such that $\psi_\alpha \circ \pi_i = \psi_i|_{U_\alpha}$, and by the uniqueness part of the affine case, these maps coincide on the open affines $U_{\alpha\beta\gamma}$ and thefore on the intersections $U_{\alpha\beta}$. They can thus be patched together to a map $\psi\colon Z \to X$, which is is unique since the ψ_α's are unique. ❏

Problem 0.5. Let A_1 and A_2 be to B-algebras. Show that one has a canonical isomorphism of functors $h_{A_1} \times h_{A_2} \simeq h_{A_1 \otimes_B A_2}$, where h_A stands for $\operatorname{Hom}_B(A, \star)$. ✷

Problem 0.6. Assume that A is an B-algebra with the property that $\operatorname{Hom}_B(A, C)$ either is empty or a singleton whatever the B algebra C is. Show that the canonical map $A \otimes_B A \to A$ sending $a \otimes a'$ to the product aa' is an isomorphism. ✷

Problem 0.7. Let A be a B-algebra and let $S \subseteq A$ be a multiplicative system. Show that $A_S \otimes_B A_S \simeq A_S$. ✷

Problem 0.8. Assume that A is an B-algebra. Given two multiplicative systems $S \subseteq A$ and $S' \subseteq A$. Show that $A_S \otimes_B A_{S'} = A_{SS'}$ where SS' denotes the multiplicative system consisting of all products ss' with $s \in S$ and $s' \in S'$. ✷

Problem 0.9. Let k be a field and x and y two variables. Describe $k(x) \otimes_k k(y)$. ✷

A useful lemma

Lemma 0.1 *If $X \times_S Y$ exists and $U \subseteq X$ is an open subscheme, then $U \times_S Y$ exists and is (canonicaly isomorphic to) an open subset of $X \times_S Y$ and projections restrict to projections. Indeed $\pi_X^{-1}(U)$ with the two restrictions $\pi_Y|_{\pi_X^{-1}(U)}$ and $\pi_X|_{\pi_X^{-1}(U)}$ as projections is a fiber product.*

Proof: Displayed the situation appears like

$$\begin{tikzcd} & Y \\ \pi_X^{-1}(U) \arrow[r, hook] \arrow[d] \arrow[ur] & X \times_s Y \arrow[u, "\pi_Y"'] \arrow[d, "\pi_X"] \\ U \arrow[r, hook] & X, \end{tikzcd}$$

and we are to verify that $\pi_X^{-1}(U)$ together with the restriction of the two projections to $\pi_X^{-1}(U)$ satisfy the universal property. If Z is a scheme and $\phi_X \colon Z \to U$ and $\phi_Y \colon Z \to Y$ are two morphisms over S we may consider ϕ_U as a map into X, and therefore they induce a map of schemes $\phi\colon Z \to X \times_S Y$ whist $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. Clearly $\pi_X \circ \phi = \phi_U$ takes values in U and therefore ϕ takes values in $\pi_X^{-1}(U)$. It follows immediately that ϕ is unique (see the exercise below), and we are through ❑

Problem 0.10. Assume that $U \subseteq X$ is an open subscheme and let $\iota\colon U \to X$ be the inclusion map. Let ϕ_1 and ϕ_2 be two maps of a schemes from a scheme Z to U and assume that $\iota \circ \phi_1 = \iota \circ \phi_2$. Then $\phi_1 = \phi_2$. ✷

When identifying $\pi_X^{-1}(U)$ with $U \times_S Y$, the inclusion map $\pi_X^{-1}(U) \subseteq X \times_S Y$ will correspond to the map $\iota \times \mathrm{id}_Y$ where $\iota\colon U \to X$ is the inclusion, so a reformulation of the lemma is that open immersions stay open immersion under change of basis.

The glueing process

The following proposition will be basis for all glueing necessay fro the construction:

Proposition 0.2 *Let $\psi_X\colon X \to S$ and $\psi_Y\colon Y \to S$ be two maps of schemes, and assume that there is an open covering $\{U_i\}_{i\in I}$ of X such that $U_i \times_S Y$ exist for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open covering of $X \times_S Y$ and projections restrict to projections.*

Proof: We need some notation. Let $U_{ij} = U_i \cap U_j$ be the intersections of the U_i's,and let $\pi_i\colon U_i \times_S Y \to U_i$ denote the projections. By lemma 0.1 there are isomorphisms $\theta_{ji}\colon \pi_i^{-1}(U_{ij}) \to U_{ij} \times_S Y$, and glueing functions we shal use $\tau_{ji} = \theta_{ij}^{-1} \circ \theta_{ji}$ that identifies $\pi_i^{-1}(U_{ij})$ with $\pi_j^{-1}(U_{ij})$. The picture is like this

$$U_i \times_S Y \supseteq \pi_i^{-1}(U_{ij}) \xrightarrow[\simeq]{\theta_{ji}} U_{ij} \times_S Y \xrightarrow[\simeq]{\theta_{ij}^{-1}} \pi_j^{-1}(U_{ji}) \subseteq U_j \times_S Y.$$

The glueing maps τ_{ij} clearly satisfy the glueing conditions being compositions of that the particular form, and the scheme emerging from glueing process is $X \times_S Y$.

The two projections are essential parts of product. The projection onto Y is there all the time since we never touch Y during the construction. The projection onto X

is obtained by glueing the projections π_i along the $\pi_i^{-1}(U_{ij})$. By lemma 0.1 we know that the when we identify $\pi_i^{-1}(U_{ij})$ as the product $U_{ij} \times_S Y$ the projection π_{ij} onto U_{ij} corresponds to the restriction $\pi_i|_{\pi_i^{-1}(U_{ij})}$. This means that $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_{ij} \circ \theta_{ji}$. To say that $\pi_i|_{\pi_i^{-1}(U_{ij})}$ and $\pi_j|_{\pi_j^{-1}(U_{ij})}$ becomes equal after glueing is to say that $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji}$ (remember that in the glueing process we identify points x and $\tau_{ji}(x)$), but this holds true since

$$\pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \theta_{ij}^{-1} \circ \theta_{ji} = \pi_{ij} \circ \theta_{ji} = \pi_i|_{\pi_i^{-1}(U_{ij})},$$

and we can glue the π_i's together to obtain π_X.

It is a matter of easy verification that the the glued scheme with the two projection has the universal property.

❑

It is worth while commenting that the product $X \times_S Y$ is not defined as a particular scheme, it is just an isomorphism class of schemes (having the fundamental property that there is a unique isomorphism respecting the projections between any two). In the proof above both $\pi_i^{-1}(U_{ij})$ and $\pi_j^{-1}(U_{ij})$ are products of $U_{ij} \times_S Y$, but the are not equal only canonically isomorphic. In the construction we could use any of them, or as we in fact did, we can use any non-specified representative in the isomorphism class. This makes the situation much symmetric in i and j.

An immediate consequence of the glueing proposition 0.2 is the following lemma, that is the case when S is affine.

Lemma 0.2 *Assume that S is affine, then $X \times_S Y$ exists*

PROOF: First if Y as well is affine, we are done. Ideed, cover X by open affine sets U_i. Then $U_i \times_S Y$ exists by the affine case, and we are in the position to apply proposition 0.2 above. We then cover Y by affine open sets V_i. As we just verified, $X \times_S V_i$ all exists and applying proposition 0.2 once more, we can conclude that $X \times_S Y$ exists. apply the glueing ❑

The final reduction

Let $\{S_i\}$ be an open affine covering of S and let $U_i = \psi^{-1}(S_i)$ and $V_i = \psi_Y^{-1}(S_i)$. By lemma 0.2 the products $U_i \times_{S_i} V_i$ all exists. Using the following lemma and for the third time the glueing proposition 0.2 we are trough:

Lemma 0.3 *With current notation, we have the equality $U_i \times_{S_i} V_i = U_i \times_S Y$.*

PROOF: The key diagram is

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ U_i \arrow[dr, "\psi_X|_{U_i}"'] & & Y \arrow[dl, "\psi_Y"] & \supseteq V_i \\ & S & & \end{tikzcd}$$

where f and g are given maps. If one follows the left path in the diagram, one ends up in S_i, and hence the same must hold following the right path. But then, V_i being equal the inverse image $\psi_Y^{-1}(S_i)$, it follows that g necessarily factors through V_i, and we are done. ❑

Notation.

If $S = \operatorname{Spec} A$ on often writes $X \times_A Y$ in short for $X \times_{\operatorname{Spec} A} Y$. If $S = \operatorname{Spec} \mathbb{Z}$, one writes $X \times Y$. In case $Y = \operatorname{Spec} B$ the shorthand notation $X \otimes_A B$ is frequently seen as well—it avoids writing Spec twice.

Diagrams arising from fiber products are frequently called *cartesian diagram* (kartesiske diagrammer); that is, the diagram

$$\begin{tikzcd} Z \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "\psi_X"] \\ Y \arrow[r, "\psi_Y"'] & S \end{tikzcd}$$

is said to be a *cartesian diagram* if there is an isomorphism $Z \simeq X \times_S Y$ with π_1 and π_2 corresponding to the two projections.

PROBLEM 0.11. Let X, Y and Z be three schemes over S. Show that $X \times_S S = X$, that $X \times_S Y \simeq Y \times_S X$ and that $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$. If T is a scheme over S, show that $X \times_S T \times_T Y \simeq X \times_S Y$. ✷

PROBLEM 0.12. Show by using the univeral property that if $\phi\colon X' \to X$ and $g\colon Y' \to Y$ are morphisms over S, then there is morphism $f \times g\colon X' \times_S Y' \to X \times_S Y$ such that

$$\begin{tikzcd} X' \times_S Y' \arrow[r, "f \times g"] \arrow[d, "\pi_{X'}"'] & X \times_S Y \arrow[d, "\pi_X"] \\ X' \arrow[r] & X \end{tikzcd}$$

and a corresponding diagram involving Y and Y' commute ✷

Examples

VARIETIES VERSUS SCHEMES In the important case that X and Y are integral schemes of finite type over the algebraically closed field k the product of the two as varities coincides with their product as schemes over k, with the usual interpretation that the varity assosiated to the scheme X is the set closed points $X(k)$ with induced topology.

The product $X \times_k Y$ will be a variety (*i.e.,* an integral scheme of finite type over k) and the closed points of the product $X \times_k Y$ will be the direct product of the closed points in X and Y; indeed, on the level of functors $h_{X\times_k Y}(k)$ equals the product $h_X(k) \times h_Y(k)$, and closed points correspond to maps of schemes $\operatorname{Spec} k \to X$.

It is not obvious that $A \otimes_k B$ is an integral domain when A and B are, and in fact, in general, even if k is a field, it is by no means true. But it holds true whenever A and B are of finite type over k and k is an algebraically closed field. The standard reference for this is Zariski and Samuel's book *Commutative algebra I* which is the Old Covenant for algebraists. It is also implicit in Hartshorn's book, exercise **3.15** b) on page 22.

However that the tensor product $A \otimes_k B$ is of finite type over when A and B are, is straight forward. If $u_1, \dots, u_m$ generate A over k and $v_1, \dots, v_m$ generate B over k the products $u_i \otimes 1$ and $1 \otimes v_j$ generate $A \otimes_k B$.

NON ALGEBRAICALLY CLOSED FIELD This case the situation is more subtle when one works over fields that are not algebraically closed. To illustrate some of the phenomena that can occure, we study a few basic examples.

EXAMPLE 0.1. A simple but illustrative example is the product $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. This scheme has *two* distinct closed points, and it is not integral—it is not even connected!

The example also shows that the underlying set of the fiber product is not necessarily equal to the fiber product of the underlying sets, although this was true for varieties over an algebraically closed field. In the present case the three schems inolved all have just one element and the their fibre product has just one point. So we issue warnings: The product of integral schemes is in general not necessarily integral! The underlying set of the fiber product is not always the fiber product of the underlying sets.

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is in fact isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$ of two copies of the complex field $\mathbb{C}$; indeed, we compute using that $\mathbb{C} = \mathbb{R}[t]/(t^2+1)$ and find

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2+1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2+1) = \mathbb{C}[t]/(t-i)(t+i) = \mathbb{C} \times \mathbb{C}$$

where for the last equation we use the Chinese remainder theorem and that the rings $\mathbb{C}[t]/(t \pm i)$ both are isomorphic to $\mathbb{C}$. equation. ❄

EXAMPLE 0.2. This little example can easily be generalize: Assume that L is a simple, separable field extension of K of degree d; that is $L = K(\alpha)$ where the minimal polynomial $f(t)$ of α over K is separable and of degree d. Let Ω be a field extension of K in which the polynomial $f(t)$ splits completely—*e.g.,* a normal extension of L or any algebraically closed field containing K— then by an argument completely analogous to the one above one finds that $L \otimes_K \Omega = \Omega \times \ldots \times \Omega$ where the product has d factors. Consequently the product scheme $\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} \Omega$ has an underlying set with d points, even if the three sets of departure all are prime spectra of fields and thus singletons.

One may push this further and construct examples where $\operatorname{Spec} K \otimes_{\operatorname{Spec} k} \operatorname{Spec} \Omega$ is not even noetherian and has infinity many points! ❄

PROBLEM 0.13. With the assumptions of the example above, check the statement that $L \otimes_K \Omega \simeq \Omega \times \ldots \times \Omega$, the product having d factors. ✷

PROBLEM 0.14. Assume that A is an algebra over the field k having a countable set $\{e_1, e_2, \ldots, e_i, \ldots\}$ of mutually orthogonal idempotents, *i.e.*, $e_i e_j = 0$ if $i \neq i$ and $e_i e_i = 1$, and assume that $e_i A \simeq k$. Assume also that every element is a finite linear combination of the e_i's.

a) Show that the ideal I_j generated by the e_i's with $i \neq j$ is a maximal ideal.

✷

EXAMPLE 0.3. In this example we let $L \in \mathbb{C}[x, y]$ be a linear form that is not real, for example $L = x + iy + 1$, and we introduce the real algebra $A = \mathbb{R}[x, y]/(L\overline{L})$. The product $L\overline{L}$ of L and its complex conjugate is a real irreducible quadric; which in the concrete example is $(x + 1)^2 + y^2$. The prime spectrum $\operatorname{Spec} A$ is therefore an integral scheme. However, the fiber product $\operatorname{Spec} A \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$ is not irreducible being the union of the two conjugate lines $L = 0$ and $\overline{L} = 0$ in $\operatorname{Spec} \mathbb{C}[x, y]$.

The scheme $\operatorname{Spec} A$ has just one real point, namely the point $(-1, 0)$ (*i.e.*, corresponding to the maximal ideal $(x + 1, y)$). The $\mathbb{C}$-points however, are plentiful.

They are contained in the $\mathbb{C}$-points $\mathbb{A}^2_{\mathbb{R}}(\mathbb{C})$, which are of orbits $\{(a, b), (\overline{a}, \overline{b})\}$ of the complex conjugation with (a, b) non-real, and form the subset of those (a, b) such that $L(a, b) = 0$. ❄

EXAMPLE 0.4. Another example along same lines as example 0.2 shows that the fiber product $X \times_S Y$ is not necessarily reduced even if both X and Y are; the point being to use an inseparable polynomial $f(t)$ in stead of a the separable one in 0.2. Let k be a non-perfect field in characteristic p which means that there is an element $a \in k$ not being a p-th power of any element in k. Let L be teh field extension $L = k(b)$ where $b^p = a$. That is, $L = k[t]/(t^p - a)$. which is a field since $t^p - a$ is an irreducible polynomial over k. However, upon being tensorized by itself over k, it takes the shape

$$L \otimes_k L = L[t]/(t^p - a) = L[t]/(t^p - b^p) = L[t]/((t - b)^p)$$

which is not reduced, the non-zero element $t - b$ being nilpotent. So we issue a third warning: the fiber product of integral schemes is not in general reduced! ❄

One can elaborate these example and construct an example of two noetherian schemes X and Y such that $X \times_S Y$ is not noetherian, even if S is the spectrum of a field. The next examples shows that if X, Y and S are fields, the product $X \times_S Y$ can even have an uncountable number of elements!

EXAMPLE 0.5. In this example we take L to the subfield of the field $\overline{\mathbb{Q}}$ of algebraic numbers that is generated by all the elements a such that $a^{2^n} = 2$ for some n. The field L is the union of the ascending chain of fields

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt[8]{2}) \subseteq \ldots \subseteq \mathbb{Q}(\sqrt[2^r]{2}) \subseteq \ldots \subseteq L \subseteq \overline{\mathbb{Q}}$$

We denote r-th field in the chain $\mathbb{Q}(\sqrt[2^r]{2})$ by L_r. The next field L_{r+1} is the quadratic extension of L_r obtained simply by adjoining $\sqrt[2^{r+1}]{2}$, or, in other words, the square root of $\sqrt[2^r]{2}$.

We let $A_r = L_r \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. By the arguments in example xxx A_r this is a finite algebra of rank 2^r over $\mathbb{Q}$. It has fine-structure induced by the smaller algebras A_s for $s \leq r$ which are all subalgebras of A_r. the algebra A_r splits as the direct product of two copies of A_{r-1}; indeed, one has $L_r \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = L_r \otimes_{L_{r-1}} \overline{\mathbb{Q}} \otimes_{\overline{\mathbb{Q}}} L_{r-1} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} = A_{r-1} \oplus A_{r-1}$ since $L_r \otimes_{L_{r-1}} \overline{\mathbb{Q}} \simeq \overline{\mathbb{Q}} \oplus \overline{\mathbb{Q}}$.

The two idempotents in A_r that induce this splitting are denoted by $e_{r,0}$ and $e_{r,1}$. They are orthogonal and their sum equals one. Each of the two subalgebras $e_{r,\epsilon}A_r$ are isomorphic to A_{r-1} with $1 \in A_{r-1}$ corresponding to $e_{r,\epsilon}$, they contain the idempotens $e_{r-1,\epsilon}$ which wil correspond to the product $e_{r,\epsilon}e_{r-1,\epsilon'}$ in A_r. Working our way down in A_r, this yields a sequence of idempotents $e_I = e_{r,\epsilon_r}e_{r-1,\epsilon_{r-1}} \ldots e_{1,\epsilon_1}$ where $I = (\epsilon_1, \ldots, \epsilon_r)$ is a sequence of 1's and 0's.

Take any sequence $\sigma = (\sigma_i)i \in \mathbb{N}$. Let $I \subseteq A$ be generted by the e_{i,ϵ_i} with $\epsilon_i \notin \sigma$. Then I is maximal. It contains all product exept $\prod_i e_{i,\sigma_i}$. And these all generate the same copy of $\overline{\mathbb{Q}} \subseteq A$!

splits as the direct product of two copies of L_n, *i.e.,* there are two orthogonal idenpotens idenpotens e and e'.

Then of course L being a field is noetherian as is $\overline{\mathbb{Q}}$, but $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not! Indeed, $\mathbb{Q}(\sqrt[2^r]{2}) \otimes \overline{\mathbb{Q}}$ is isomorphic to the direct product of 2^r-copies of $\overline{\mathbb{Q}}$ so $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is the union of a sequence of subrings each being a direct product of a steadily increasing number of copies of $\overline{\mathbb{Q}}$. ❄

PROBLEM 0.15. Show in detail that $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not noetherian. ✱

Base change

The fiber product is in constant use in algebraic geometry, and it is an astonishingly versatile and flexible instrument. In different situations it serves quite different purposes and appears under different names. We shall comment on some of the most frequently encountered applications, and we begin with notion of base change.

In its simples and earliest appearances base change is just extending the field over which one works; *e.g.,* in Galois theory, or even in the theory of real polynomals, when studying an equation with coefficients in a field k one often finds it fruitful to study the equation over a bigger field K. Generalizing this to extensions of algebras over which one works, and then to schemes, one arrives naturally at the fiber product.

If X is a scheme over S and $T \to S$ is map. Considering $T \to S$ as change of base schemes one frequently writes $X_T = X \times_S T$ and says that X_T is obtained from X by *base change* (basisforandring) or frequently that X_T is the *pull back* (tilbaketrekningen) of X along $T \to S$. This is a functorial construction, since if $\phi\colon X \to Y$ is a morphism over S, there is induced a morphism $\phi_T = \phi \times \mathrm{id}_T$ from X_T to Y_T over T, and one easily checks that $(\phi \circ \phi')_T = \phi_T \circ \phi'_T$. The defining properties of ϕ_T are $\pi_Y \circ \phi_T = \phi \circ \pi_X$ and $\pi_T \circ \phi = \pi_T$, as depicted in the diagram:

$$\begin{tikzcd}
 & T \\
X_T = X \times_S T \arrow[ur, "\pi_T"] \arrow[r, "\phi_T"] \arrow[dr, "\phi\circ\pi_X"'] & Y \times_S T \arrow[u, "\pi_T"'] \arrow[d, "\pi_Y"] \\
 & Y.
\end{tikzcd}$$

Problem 0.16. Verify in detail that $(\phi \circ \phi')_T = \phi_T \circ \phi'_T$. ✷

If $\mathcal{P}$ is a property of morphisms, one says that $\mathcal{P}$ is stable under base change if for any T over S, the map f_T has the property $\mathcal{P}$ whenever f has it. For example, another way of phrasing lemma 0.1 on page 5 is to say that being an open immersion is stable under base change.

Geometric points

If X is a scheme a *geoemrtric point* (et geometrisk punkt) consists of an algebraically closed field k and a morphism $\operatorname{Spec} k \to X$. Giving such a geowmeyric point is equivalent to give a point $x \in X$ and a field extension $k(x) \subseteq k$.

For example

Example 0.6. One has $\operatorname{Spec}\mathbb{Q} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Q} = \operatorname{Spec}\mathbb{Q}$. Indeed, there is only one ring homomorphism from $\mathbb{Q}$ to any ring, so $h_{\operatorname{Spec}\mathbb{Q}}(T)$ is either a singleton or empty. It follows that the fibre product $h_{\operatorname{Spec}\mathbb{Q}}(T) \times_{\mathbb{Z}} h_{\operatorname{Spec}\mathbb{Q}}(T)$ either is a singleton or empty, hence OK. ❄

Problem 0.17. Let p and q be two different primes. Show that $\operatorname{Spec}\mathbb{F}_p \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_q = \emptyset$. ✷

Problem 0.18. Show that if $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are affine schemes of finite type over a field k, then $\operatorname{Spec} A \times_k \operatorname{Spec} B$ is non-empty. Is the same true if one of them is not of finite type? Hint: Yes, *e.g.*, show OK if A and B are field extensions of k. ✷

Problem 0.19. Recall that $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ in the prime ideal (p) generated by p. Show that $\operatorname{Spec}\mathbb{Z}_{(p)} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}_{(p)} = \operatorname{Spec}\mathbb{Z}_{(p)}$. ✷

PROBLEM 0.20. Assume that p and q are two different primes. Show that $\operatorname{Spec}\mathbb{Z}_{(p)} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}_{(q)} = \operatorname{Spec}\mathbb{Q}$. ✷

PROBLEM 0.21. Let X be the scheme obtained by glueing $X_2 = \operatorname{Spec}\mathbb{Z}_{(p)}$ to it self along the generic point. Show that $X \times_{\mathbb{Z}} X$ is obtained by glueing four copies of X_2 together along the generic points. Show that the diagonal $\Delta \subseteq X \times_{\operatorname{Spec}\mathbb{Z}} X$ is the glueing of two of them and therefore is not closed. ✷

Scheme theoretical fibres

In most parts of mathematics, when one studies a map of some sort, a knowledge of what the fibres of the map are, is of great help. So also in the theory of schemes.

Suppose that $\phi\colon X \to Y$ is a map of schemes and that $y \in Y$ is a point. We are aim at giving a scheme theoretical definition of the fiber $\phi^{-1}(y)$; and having the fiber product at our disposal, nothing is more natural than defining the fiber to be the fiber product $\phi_{-1}(Y) = \operatorname{Spec} k(y) \times_Y X$, where $\operatorname{Spec} k(y) \to Y$ is the map corresponding to the point y. Recall that $k(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$ and the map is the composition $\operatorname{Spec} k(y) \to \operatorname{Spec}\mathcal{O}_{Y,y} \to Y$. For diagrammoholics, the scheme theoretical fiber of ϕ over y fits into the cartesian diagram

$$\begin{tikzcd} \phi^{-1}(y) = X \times_Y \operatorname{Spec} k(x) \arrow[r] \arrow[d] & X \arrow[d, "\phi"] \\ \operatorname{Spec} k(x) \arrow[r] & Y \end{tikzcd} .$$

As the next lemma will show, the underlying topological space of $\phi^{-1}(y)$ is the topological fiber, but in addition there is a scheme structure on it. In in many cases it is not reduced, and this a mostly a good thing since it makes certain continuity results true.

Proposition 0.3 *The inclusion $X_y \to X$ of the scheme theoretical fiber is a homeomorphism onto the topological fiber $\phi^{-1}(y)$.*

PROOF: We start with the affine case, obviously Y can always without loss of generality be assumed to be affine, say $Y = \operatorname{Spec} B$, but to begin with we adopt the additional assumption that X be affine as well, let's say $X = \operatorname{Spec} A$.

The map ϕ of affine schemes is induced by a map of rings $\alpha\colon B \to A$. Let $\mathfrak{p} \subseteq B$ be a prime ideal. We have the following equality between sets

$$\{\, \mathfrak{q} \subseteq A \mid \mathfrak{q} \text{ prime ideal }, \alpha^{-1}(\mathfrak{q}) \supseteq \mathfrak{p} \,\} = \{\, \mathfrak{q} \subseteq A \mid \mathfrak{q} \text{ prime ideal }, \mathfrak{q} \supseteq \mathfrak{p}A \,\}.$$

In the particular case that $\mathfrak{p}$ is a maximal ideal, the inclusion $\alpha^{-1}(\mathfrak{q}) \supseteq \mathfrak{q}$ is necessarily an equality, and the sets above describe the fiber set-theoretically:

$$\phi^{-1}(\mathfrak{p}) = \{\, \mathfrak{q} \subseteq A \mid \mathfrak{q} \supseteq \mathfrak{p}A \,\} \simeq \operatorname{Spec} A/\mathfrak{q}A.$$

But this also describes the good old embedding of Spec $A/\mathfrak{q}A$ into Spec A identifying it with the closed subscheme $V(\mathfrak{q}A)$, and therefore this yields a homeomorphism between Spec $A/\mathfrak{p}A$ and the topological fiber $\phi^{-1}(\mathfrak{p})$. On the other hand by standard equalities between tensor products one has

$$A/\mathfrak{p}A = A \otimes_B B/\mathfrak{p}B = A \otimes_B k(y),$$

and so the scheme theoretical fiber $\phi^{-1}(y) = X_y = X \times_Y \operatorname{Spec} k(y) = \operatorname{Spec} A \otimes_B k(y)$ is in a canonical way homeomorphic to the topological fiber.

If $\mathfrak{p}$ is not a maximal ideal, the set Spec $A/\mathfrak{p}A$ is strictly bigger than the fiber, the superfluous prime ideals being those for which $\alpha^{-1}\mathfrak{q}$ strictly bigger than $\mathfrak{p}$. When localizing in the multiplicative system $S = B \setminus \mathfrak{p} \subseteq B$, these superfluous prime ideals go non-proper, since they all contain elements of the form $\alpha(s)$ with $s \in S$. Hence the points in the fiber correspond to the primes in the localized ring $(A/\mathfrak{p}A)_\mathfrak{p}$. Standard formulas for the tensor product gives on the other hand the equality

$$(A/\mathfrak{p}A)_\mathfrak{p} = A \otimes_B B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = A \otimes_B k(y).$$

The topologies coincides as well, since $\operatorname{Spec}(A/\mathfrak{p}A)_\mathfrak{p}$ naturally is a subscheme of Spec $A/\mathfrak{p}A$; induced topology being the one a prime spectrum.

In the general case, *i.e.,* when X is no longer affine, we cover X by open, affine U_i's. By lemma xxx, displayed in slight rotated version below, we know that $U \cap X_s = U_s$.This shows that the scheme theoretical and the topological fiber coincide as topological spaces.

$$\begin{tikzcd}
U_s \arrow[r] \arrow[d] & U \arrow[d] \\
X_s \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} k(y) \arrow[r] & Y
\end{tikzcd}$$

❑

Example 0.7. We take a look at a simple but classic example: The map Spec $k[x,y]/(x-y^2) \to \operatorname{Spec} k[x]$ induced by the inclusion of $B = k[x] \subseteq k[x,y]/(x-y^2) = A$. Geometrically one would say it is just the projection of the parabola onto the x-axis.

If $a \in k$ computing the fiber yields, where $k(a)$ denotes the field $k(a) = k[t]/(t-a)$ (which of course is just a copy of k).

$$k[x,y]/(x-y^2) \otimes_k k(a) = k[y]/(y^2 - a).$$

Several cases can occur, apart from the characteristic two case being special.

- If a does not have a square root in k, the fiber is Spec $k(\sqrt{a})$ where $k(\sqrt{a})$ is a quadratic field extension of k.

- ☐ In case a has a square root in K, way $b^2 = a$, the polynomial $y^2 - a$ factors as $(y-b)(y+b)$, and the fiber becomes $\operatorname{Spec} k[y]/(y-b) \times \operatorname{Spec} k[y]/(y+b)$, the disjoint union of two copies of $\operatorname{Spec} k$

- ☐ Finally, the case appears when $a = 0$. The the fiber is not reduced, but equals $\operatorname{Spec} k[y]/y^2$.

We also notice that the generic fiber is the quadratic extension $k(x)(\sqrt{x})$ of the function field $k(x)$.

Over perfect fields k of characteristic two, the picture is completely different. Then a is a square, say $a = b^2$ and as $(y^2 - b^2) = (y-b)^2$ non of the fibers are reduced, they equal $\operatorname{Spec} k[y]/(y-b)^2$, except the generic one which is $k(x)(\sqrt{x})$. One observes interestingly enough, that all the non-reduced fibers deform into a field!

PROBLEM 0.22. Discuss what happens if the field k i not perfect. ✷

❄

The diagonal

Let X/S be a scheme over S. There is a canonical map $\Delta_{X/S} \colon X \to X \times_S X$ of schemes over S called the *diagonal map* or the *diagonal morphism* (diagonalavbildningen eller diagonalmorfismen). The two components maps of Δ are both equal to the identity id_X; that is, the defining properties of $\Delta_{X/S}$ are $\pi_i \circ \Delta_{X/S} = \mathrm{id}_X$ for $i = 1, 2$ where the π_i's denote the two projections. On the level of functors the diagonal map is simply what we understand with the diagonal. It is the map

$$h_{X/S}(T/S) \to h_{X/S}(T/S) \times h_{X/S}(T/S)$$

that sends a T-point ξ to the pair (ξ, ξ). If T/S is a scheme, a T-point in $X \times_S X$ takes values in the diagonal—that is, factors via $\Delta_{X/S}$—if and only if the two component-maps coincide.

In the case X and S are affine schemes, the diagonal has a simple and natural interpretation in terms of algebras; it corresponds to most natural map, the multiplication map:

$$\mu \colon A \otimes_B A \to A.$$

It sends $a \otimes a'$ to the product aa' and then extends to $A \otimes_B A$ by linearity. The projections correspond to the two maps $\iota_i \colon A \to A \otimes_B A$ sending a to $a \otimes 1$ respectively to $1 \otimes a$. Clearly it holds that $\mu \circ \iota_i = \mathrm{id}_A$, and on the level of schemes this translates into the defining relations for diagonal map. We have established the following:

Proposition 0.4 *If X an affine scheme over the affine scheme S, then the diagonal $\Delta_{X/S} \colon X \to X \times_S X$ is a closed imbedding.*

This is not generally true for schemes and shortly we shall give examples, however from the lemma 0.4 we justed proved, it follows readily that the image $\Delta_{X/S}(X)$ is locally closed—*i.e.,* the diagonal is locally a closed embedding:

Proposition 0.5 *The diagonal $\Delta_{X/S}$ is locally a closed embedding.*

PROOF: Begin with covering S by open affine subset and subsequently cover each of their inverse images in X by open affines as well. In this way one obtains a covering of X by affine open subsets U_i whose images in S are contained in affine open subsets S_i. The products $U_i \times_{S_i} U_i = U_i \times_S U_i$ are open and affine, and their union is an open subset containing the image of the diagonal. By proposition 0.4 above the diagonal restricts to a closed embedding of U_i in $U_i \times_{S_i} U_i$. ❏

PROBLEM 0.23. In the setting of the previous proof, show that $\Delta_{X/S}|_{U_i} = \Delta_{U_i/S}$. ✸

On says that the scheme X/S is *separated* (separert) over S, or that the structure map $X \to S$ is separated if the diagonal map is a closed imbedding. If X is separated over $\operatorname{Spec} \mathbb{Z}$ one says for short that it is separated.

Assume we are given two maps $\phi_i \colon Z \to X$, with $i = 1, 2$, of schemes over S. Let $\phi \colon Z \to X \times_S X$ be the map whose components are ϕ_1 and ϕ_2, ; that is, the map whose defining relations are $\pi_i \circ \phi = \phi_i$. One defines the *equalizer* of the two maps $E(\phi_1, \phi_2)$ as the inverse image of the diagonal map; that is, there is cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & X \times_S X \\ {\scriptstyle E}\uparrow & & \uparrow{\scriptstyle \Delta_{X/S}} \\ E(\phi_1, \phi_2) & \longrightarrow & X. \end{array}$$

The scheme $E(\phi_1, \phi_2)$ has the following universal property.

Lemma 0.4 *Given two maps $\phi_i \colon Z \to X$ and let $E(\phi_1, \phi_2)$ be the equaliser. A map $\psi \colon T \to Z$ satisfies $\phi_1 \circ \psi = \phi_2 \circ \psi$ if and only if ψ factors through $E(\phi_1, \phi_2)$.*

The equalizer $E(\phi_1, \phi_2)$ is therefore, in some sense, the largest subschemes of Z over which the two maps ϕ_1 and ϕ_2 coincide. Some authors call is the *kernel* of the two maps ϕ_1 and ϕ_2, and is also known as the *scheme of coincidence* of the two maps.

The notion of equalizers has a meaning in any category, but of course its unrestricted existence is a feature of the category of schemes (however, sheared with a lot of other categories. If fiber products unconditionally exist, equalizers do as well).

Assume that C is the category and that $\phi_1, \phi_2 \colon Z \to X$ are two maps from C. An equalizer $E = E(\phi_1, \phi_2)$ is an object of C together with a map $\iota \colon E \to Z$ from C having the following universal property: For every map $\psi \colon T \to Z$ it holds true that $\phi_1 \circ \psi = \phi_2 \circ \psi$ if and only if there is a unique map $\eta \colon T \to E$ in C such that $\psi = \iota \circ \eta$; that is, if and only if ψ factors through E. In this context the content of lemma 0.4

is that the pull back of the diagonal along ϕ is the equalizer in the category Sch/S of schemes over S.

PROOF OF LEMMA 0.4: The lemma is obvious for $Z = X \times_S X$ and $\phi_i = \pi_i$; that is, the equaliser $E(\pi_1, \pi_2)$ of the two projections is the diagonal $\Delta_{X/S}$. Indeed, if $\psi\colon Z \to X \times_S X$ is a map of S-schemes with component maps ψ_i that factors through the diagonal, clearly the two maps $\pi_i \circ \psi = \psi_i$ are equal since both projections satisfy $\pi_i \circ \Delta_{X/S} = \mathrm{id}_X$. It is equally obvious that if $\psi_1 = \psi_2$ the map ψ factors through the diagonal, *i.e.*, one has, say, $\psi_i = \Delta_{X/S} \circ \psi_1$, and maps into the product being determined by the components, it holds that $\psi = \Delta_{X/S} \circ \psi_1$.

In the general setting, if ψ is a T-point of Z one has $\pi_i \circ \phi \circ \psi = \phi_i \circ \psi$. Hence after what we just did, the composition $\phi \circ \psi$ factors through the diagonal—meaning that ψ factors through the inverse image of the diagonal—if and only if $\phi_1 \circ \psi = \phi_2 \circ \psi$. ❏

Proposition 0.6 *The scheme X/S is separated if and only the equalizers of any pair of maps of S-schemes $\phi_1, \phi_2\colon Z \to X$ is closed.*

PROOF: If all equalizers are closed, the diagonal is closed being the equalizer of the two projections. If the diagonal is closed any equalizer is, being the inverse image of the diagonal. ❏

EXAMPLE 0.8. The simplest schemes that are not separated are obtained by glueing the prime spectrum of a discrete valuation ring to itself along the generic point.

To be precise, let R be the DVR with fraction field K. Then $\operatorname{Spec} R = \{x, \eta\}$ where x is the closed point corresponding to the maximal ideal, and η is the generic point corresponding to the zero ideal. The generic point η is an open point (the complement of $\{\eta\}$ is the closed point x) and the support of the open subscheme $\{\eta\} = \operatorname{Spec} K$. By by the glueing lemma, we may glue one copy of $\operatorname{Spec} R$ to another copy of $\operatorname{Spec} R$ by identifying the generic points—that is, the open subschemes $\operatorname{Spec} K$—in the two copies.

In this manner we contruct a sceheme Z_R togethre with two open embeddings $\iota_i\colon \operatorname{Spec} R \to Z_R$. They send the generic point η to the same point, which is an open point in Z_R, but they differ on the closed point x. Thus, the equalizer of the two embeddings is the open subscheme $\{\eta\} = \operatorname{Spec} K$. This is not a closed subscheme, and Z_R is not separated. ❈

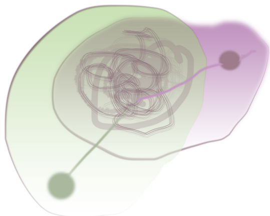
PROBLEM 0.24. Show that $Z_R \times Z_R$ is obtained by glueing *four* copies of $\operatorname{Spec} R$ together along their generic points. Show that the diagonal is open and not closed. ✱

In some sense, these tiny schemes Z_R together with some of their bigger cousins are always at the root of a non separated scheme. For any valuation ring R with maximal ideal $\mathfrak{m}_R$ one can glue two copies of $\operatorname{Spec} R$ together along some open set $U \subseteq R \setminus \{\mathfrak{m}_R\}$ to get schemes Z_R, using these schemes one has the following proposition whose proof we shall not give, it has to surmount a few technical difficulties making it not very transparent (If you are interested, try reading the proof in Hartshorn's book).

Proposition 0.7 *Assume that X is a quasi-compact scheme. If X/S is non-separated if and only if it contains a subschemes X_R for some valuation ring R.*

A more usual way of phrasing this is as follows in that guise the result is called the *valuation criterion for separateness*:

Proposition 0.8 *A quasi compact scheme X is separated if and only if the following condition is satisfied: For any valuation ring R with fraction field K, a map $\operatorname{Spec} K \to X$ over S has at most one extension to $\operatorname{Spec} R \to X$*



A tiny non-separated scheme

It is not true that maps $\operatorname{Spec} K \to X$ like in the criterion always can be extended—schemes with that property are called proper schemes—so the criterion says there can never be two different extensions in case X is separated. One does frequently encounter non-separated scheme in practice, but some very nice properties are only true for separated schemes, and this legitimates the notion. Of course, one needs good criteria to be sure we have a large class of separated schemes. We have already seen that all affine schemes are separated, and when we come to that point projective schemes will turn out to be separated as well.

One of the nice properties affine schemes enjoy, is the following:

Proposition 0.9 *Assume that X is separated and that U and V are to open affine subscheme. Then the intersection $U \cap V$ is affine and the map $\Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V) \to \Gamma(U \cap V, \mathcal{O}_{U\times V})$ is surjective*

PROOF: The product $U \times V$ is an open and affine subset of $X \times V$, and $U \cap V = \Delta_X(X) \cap (U \times V)$. So if the diagonal is closed, $U \cap V$ is a closed subset of the affine set $U \times V$ hence affine. It is a general fact about products of affine schemes that one has

$$\Gamma(U \times V, \mathcal{O}_{U\times V}) = \Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V),$$

and as $U \cap V$ is a closed subscheme of $U \times V$, the restriction map

$$\Gamma(U \times V, \mathcal{O}_{U\times V}) \to \Gamma(U \cap V, \mathcal{O}_{U\cap V})$$

is surjective. ❏