Mandatory assignment MAT4215 – Spring 2017

The assignment must be handed in in the special box at the 7. floor in Niels Henrik Abels hus before 14:30, **Thursday May 18th**. Remember to fill in and attach a front page - front pages are found near the box or online.

The problems marked with * are not mandatory. If you have any questions or comments about the problems, feel free to ask me in the lectures or email me at `johnco@math.uio.no`.

Problem 1. Give an example of a scheme X, a field K, and a morphism of ringed spaces $\operatorname{Spec} K \to X$ which is *not* a morphism of schemes.

Problem 2. Describe the scheme-theoretic fibers in all points of the following morphisms.

(a) $\operatorname{Spec} \mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec} \mathbb{C}[x]$
(b) $\operatorname{Spec} \mathbb{C}[x,y]/(xy) \to \operatorname{Spec} \mathbb{C}[x]$
(c) $\operatorname{Spec} \mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec} \mathbb{C}[x]$
(d) $\operatorname{Spec} \mathbb{Z}[x,y]/(xy^2-m) \to \operatorname{Spec} \mathbb{Z}$, where m is a non-zero integer.

Which fibers are irreducible or reduced?

Problem 3. Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\Gamma : \mathrm{Mod}_X \to \mathrm{Mod}_A$ and $\sim: \mathrm{Mod}_A \to \mathrm{Mod}_X$ are adjoint: That is, for any $M, \mathcal{F}$, there is a natural isomorphism

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \simeq \mathrm{Hom}_{\mathscr{O}_X}(\widetilde{M}, \mathcal{F})$$

Problem 4. Let X be a noetherian scheme and let $\mathcal{F}$ be a coherent sheaf. Prove the following statements:

(a) If the stalk $\mathcal{F}_x$ is a free $\mathscr{O}_{X,x}$-module, then there is a neighbourhood U of x such that $\mathcal{F}|_U$ is free.

(b) $\mathcal{F}$ is locally free if and only if the stalks $\mathcal{F}_x$ are free $\mathscr{O}_{X,x}$-modules for every $x \in X$.

Problem 5*. A scheme is *normal* if all of its local rings are integrally closed domains.

(a) Let X be an integral scheme. For each affine open $U = \operatorname{Spec} A$, let $\widetilde{U} = \operatorname{Spec} \tilde{A}$, where $\tilde{A}$ is the integral closure of A in its quotient field. Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, with a morphism $\tilde{X} \to X$. This is called the *normalization* of X.

(b) Let $X = \operatorname{Spec} A$, where

$$A = \mathbb{C}[x,y]/(x^2 - y^5).$$

Describe $\widetilde{X}$ and the map $\widetilde{X} \to X$.