# Mandatory assignment MAT4215 – Spring 2021

The assignment must be submitted electronically by **14:30, Thursday May 13th**.

You need at least 50% to pass. If you have any questions or comments about the problems, feel free to email me at `johnco@math.uio.no`.

Feel free to use whatever results from Commutative Algebra you want.

**Problem 1.** [10%]
Let $k$ be a field. Show that any morphism of schemes $\mathbb{P}^1_k$ to an affine $k$-scheme is constant.

**Problem 2.** [25%]
Describe the scheme theoretic fibers in all points of the following morphisms.

(1) $f : \operatorname{Spec} \mathbb{C}[x, y]/(xy - 1) \to \operatorname{Spec} \mathbb{C}[x]$
(2) $f : \operatorname{Spec} \mathbb{C}[x, y]/(x^2 - y^2) \to \operatorname{Spec} \mathbb{C}[x]$
(3) $f : \operatorname{Spec} \mathbb{C}[x, y]/(xy) \to \operatorname{Spec} \mathbb{C}[x]$
(4) $f : \operatorname{Spec} \mathbb{Z}[x, y]/(xy^2 - m) \to \operatorname{Spec} \mathbb{Z}$, where $m$ is a non-zero integer.

Which fibers are irreducible? Which are reduced?

**Problem 3.** [20%]
Show the following:
(i) The skyscraper sheaf of $k$ on $\mathbb{A}^1_k$ at the origin 0 is quasi-coherent.
(ii) The skyscraper sheaf of $k(T)$ on $\mathbb{A}^1_k$ at the origin 0 is *not* quasi-coherent.
(iii) If $X$ is variety over an algebraically closed field $k$, then the skyscraper sheaf of $k$ at any closed point $x \in X$ is quasi-coherent.

**Problem 4.** [25%]
Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathscr{O}_X$-module, and $\mathscr{E}$ a locally free sheaf of finite rank. Show that there is a natural isomorphism

$$f_*(\mathcal{F} \otimes f^*\mathscr{E}) \simeq f_*(\mathcal{F}) \otimes \mathscr{E}.$$

**Problem 5.** [30%]
Let $f : X \to Y$ be a morphism of schemes and let $x \in X$ be a point. We say that:

- A quasi-coherent sheaf $\mathcal{F}$ on $X$ is *flat over* $Y$ at $x$ if $\mathcal{F}_x$ is flat as a $\mathscr{O}_{Y,f(x)}$-module (where $\mathcal{F}_x$ is considered as a $\mathscr{O}_{Y,f(x)}$-module via the natural map $f^\#_x : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,f(x)}$).
- $\mathcal{F}$ is *flat* if it is flat at every point in $X$.
- $f$ is flat if $\mathscr{O}_X$ is flat over $Y$

(i) Show that open embeddings are flat. What about closed immersions?
(ii) Show that a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if the map of rings $A \to B$ is flat. More generally, a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Spec} B$ is flat over $\operatorname{Spec} A$ if and only if $M$ is flat as an $A$-module.
(iii) Which of the morphisms in Problem 2 are flat?
(iv) Prove that the blow-up morphism $\pi : Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ is not flat.