Sheaves

Recall :

Definition 1.1 *Let X be a topological space. A* presheaf of abelian groups $\mathcal{F}$ *on* X *consists of the following two sets of data:*

i) for each open $U \subseteq X$, an abelian group $\mathcal{F}(U)$;

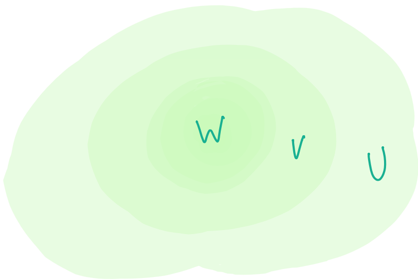
ii) for each pair of nested opens $V \subseteq U$ a group homomorphism (called restriction maps*)*

$$\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V).$$

The restriction maps must furthermore satisfy the following two conditions:

i) for any open $U \subseteq X$, one has $\rho_{UU} = id_{\mathcal{F}(U)}$;

ii) for any three nested open subsets $W \subseteq V \subseteq U$, one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.



We will usually write $s|_V$ for $\rho_{UV}(s)$ when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called 'sections' (or 'sections over U'). We will often also write $\Gamma(U, \mathcal{F})$ for the group $\mathcal{F}(U)$; here Γ is the 'global sections'-functor

Definition 1.2 *A presheaf $\mathcal{F}$ is a* sheaf *if it satisfies the two conditions:*

i) *(Locality axiom) Given an open subset $U \subseteq X$ with an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ and a section $s \in \mathcal{F}(U)$. If $s|_{U_i} = 0$ for all i, then $s = 0 \in \mathcal{F}(U)$.*

ii) *(Gluing axiom) If U and $\mathcal{U}$ are as in (i), and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections matching on the overlaps; that is, they satisfy*

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \qquad \forall i, j \in I,$$

then there exists a section $s \in \mathcal{F}(U)$ so that $s|_{U_i} = s_i$ for all i.

For each open cover $\mathcal{U} = \{U_i\}$ of an open set $U \subseteq X$ there is a sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_i \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

the maps α, β are defined by $\alpha(s) = (s|_{U_i})_i$, and $\beta(s_i) = (s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j})_{i,j}$.

⇝ Then $\mathcal{F}$ is a sheaf if and only if these sequences are exact.

$s \in \ker \alpha \iff (s|_{U_i})_i = 0$ for alle i

$\longrightarrow s = 0 \iff$ Lokalitet gjelder

$s_i \in \ker \beta \Rightarrow s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.

$\ker \beta = \operatorname{im} \alpha \iff$ Liming gjelder

Example 1.3 *(The empty set)* There is a subtle point about taking U to be the empty set in the definition of a sheaf. If $\mathcal{F}$ is a sheaf, we are forced to define $\mathcal{F}(\emptyset) = 0$. Indeed, note that the empty set is covered by the empty open covering, and the empty product is 0, so the sheaf sequence looks like $0 \to \mathcal{F}(\emptyset) \to 0$. ☆

1.2 *Morphisms between (pre)sheaves*

A *morphism* (or simply *map*) $\phi : \mathcal{F} \to \mathcal{G}$ *of (pre)sheaves* on a space X is collection of maps (i.e., group homomorphisms) $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ indexed by the open sets in X and compatible with the restriction maps:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V). \end{array}$$

In this way the sheaves of abelian groups on X form a category AbSh_Xwhose objects are the sheaves and the morphisms the maps between them.

Subsheaves and saturation

If $\mathcal{F}$ is a presheaf on X, a *subpresheaf* $\mathcal{G}$ is a presheaf such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for every open U, and such that the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, of course $\mathcal{G}$ is called a *subsheaf*.

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G} \subseteq \mathcal{F}$ a subpresheaf. We say that a section $s \in \mathcal{F}(U)$ *locally lies* in $\mathcal{G}$ if for some open covering $\{U_i\}_{i \in I}$ of U one has $s|_{U_i} \in \mathcal{G}(U_i)$ for each i.

Definition 1.4 *We define the* sheaf saturation $\overline{\mathcal{G}}$ *of* $\mathcal{G}$ *in* $\mathcal{F}$ *by letting the sections of* $\overline{\mathcal{G}}$ *over* U *be the sections of* $\mathcal{F}$ *over* U *that locally lie in* $\mathcal{G}$.

The sheaf saturation $\overline{\mathcal{G}}$ is again a subpresheaf of $\mathcal{F}$ (with restriction maps being the ones induced from $\mathcal{F}$). In fact, $\overline{\mathcal{G}}$ is, almost by definition, a sheaf.

$$\overline{\mathcal{G}}(U) = \{ s \in \mathcal{F}(U) \mid s \text{ localy lies in } \mathcal{G} \}$$

$$\mathcal{G} \subseteq \overline{\mathcal{G}} \subseteq \mathcal{F}$$

Locality: $s \in \overline{\mathcal{G}}(U)$ s.t $s|_{U_i} = 0 \Rightarrow s = 0$ in $\mathcal{F}$ $\Rightarrow$ ok

Gluing: $s_i \in \overline{\mathcal{G}}(U_i)$ s.t $s_i = s_j$ on $U_i \cap U_j$ $\Rightarrow$ s_i glue to an element $s \in \mathcal{F}$ ($\mathcal{F}$ is a sheaf). But $s|_{U_i} = s_i \in \mathcal{G}(U_i)$

$\Rightarrow$ s localy lies in $\mathcal{G}$ $\Rightarrow$ $s \in \overline{\mathcal{G}}(U)$.

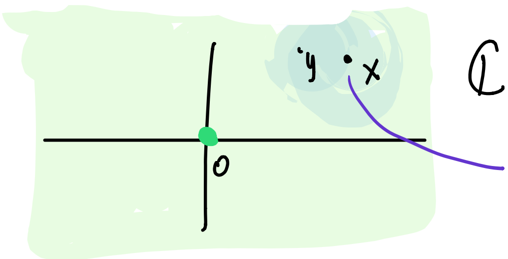
* 1.5 Take $X = \mathbb{R}^n$ and let $C(X, \mathbb{R})$ be the sheaf whose sections over an open set U is the ring of continuous real valued functions on U, and the restriction maps ρ_{UV} are just the good old restriction of functions. Then $C(X, \mathbb{R})$ is a sheaf of rings (functions can be added and multiplied), and both the sheaf axioms are satisfied. Indeed, any function $f : X \to \mathbb{R}$, which restricts to zero on an open covering of X is the zero function. Also, given continuous functions $f_i : U_i \to \mathbb{R}$ agreeing on the overlaps $U_i \cap U_j$, we can form the continuous function $f : U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ for any i such that $x \in U_i$.

1.6 For a second familiar example, let $X \subseteq \mathbb{C}$ be an open set. On X one has the sheaf $\mathscr{O}_X$ of holomorphic functions. That is, for any open $U \subseteq X$ the sections $\mathscr{O}_X(U)$ is the ring of holomorphic (i.e., complex analytic) functions on U.

1.7 *(A presheaf which is not a sheaf)* Let us continue the set-up in Example 1.6 to make another example of a presheaf which is not a sheaf. Let $X = \mathbb{C}$, and let $\mathcal{O}_X$ denote the sheaf of holomorphic functions. $\mathcal{O}_X$ contains the subpresheaf given by

$$\mathcal{F}(U) = \{ f \in \mathcal{O}_X(U) \mid f = g^2 \text{ for some } g \in \mathcal{O}_X(U) \} .$$

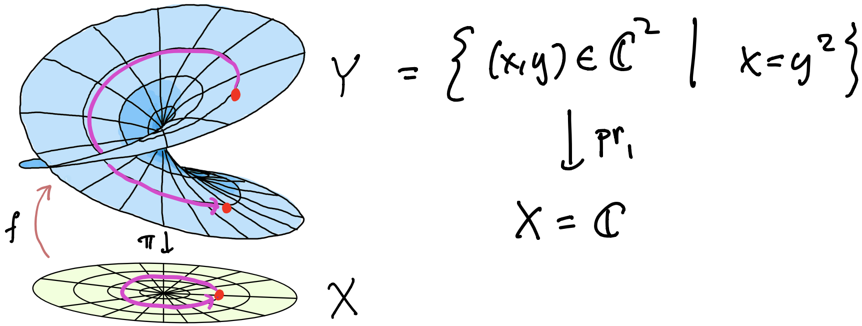
This is not a sheaf, because the Gluing axiom fails: The function $f(z) = z$ has a holomorphic square root near any point $x \in X$, but it is not possible to glue these together to a global square root function $\sqrt{z}$ on all of X.



can define a square root locally

however, Locality holds

1.8 *(Constant presheaf)* For any space X and any abelian group A one has the *constant presheaf* whose group of sections over any nonempty open set U equals A and equals 0 if $U = \varnothing$. This is not a sheaf, since if $U \cup U'$ is a disjoint union, any choice of elements $a, a' \in A$ will give sections over U and U' respectively, and they match on the intersection, which is empty. But if $a \neq a'$, they cannot be glued.



1.10 *(A Riemann surface)* Let $X = \mathbb{C}$ and $Y \subset \mathbb{C} \times \mathbb{C}$ denote the locus

$$Y = \{(x, y) \mid x = y^2\}$$

We have a map $\pi : Y \to X$ given by the first projection. Consider the presheaf on X given by

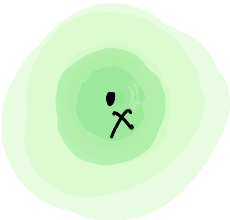
$$\mathcal{G}(U) = \{f : U \to Y \mid f \text{ is holomorphic, and } \pi \circ f = \mathrm{id}_U\}.$$

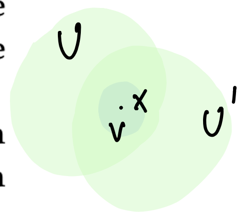
This is naturally a subpresheaf of the sheaf $C(X, Y)$, and in fact it is a sheaf.

1.12 Let V be an algebraic variety (*e.g.*, an algebraic set in $\mathbb{A}_k^n$ or $\mathbb{P}_k^n$) with the Zariski topology. For each open $U \subseteq X$, define $\mathcal{O}_V(U)$ to be the ring of regular functions $U \to k$. This is certainly a presheaf, and in fact, a sheaf. ☆

1.4 *Stalks*

Suppose we are given a presheaf $\mathcal{F}$ of abelian groups on X. With every point $x \in X$ there is an associated abelian group $\mathcal{F}_x$ called the *stalk* of $\mathcal{F}$ at x. The elements of $\mathcal{F}_x$ are called *germs of sections* near x and are designed to capture the nature of sections near the point.





The definition of $\mathcal{F}_x$ goes as follows: We begin with the *disjoint* union $\coprod_{x \in U} \mathcal{F}(U)$ whose elements we index as pairs (s, U) where U is any open neighbourhood of x and s is a section of $\mathcal{F}$ over U. We want to identify sections that coincide near x; that is, we declare (s, U) and (s', U') to be equivalent, and write $(s, U) \sim (s', U')$, if there is an open $V \subseteq U \cap U'$ with $x \in V$ such that s and s' coincide on V; that is, if one has

$$s|_V = s'|_V.$$

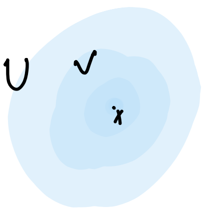
Definition 1.13 *The stalk $\mathcal{F}_x$ at $x \in X$ is by definition the set of equivalence classes*

$$\mathcal{F}_x = \coprod_{x \in U} \mathcal{F}(U) / \sim .$$

In case $\mathcal{F}$ is a sheaf of abelian groups, the stalks $\mathcal{F}_x$ are all abelian groups.

The germ of a section

For any neighbourhood U of $x \in X$, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section s to the equivalence class where the pair (s, U) belongs. This class is called the *germ* of s at x, and a common notation for it is s_x.



$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ {\scriptstyle \rho_{UV}}\downarrow & \nearrow & \\ \mathcal{F}(V). & & \end{array}$$

$\mathcal{F} \rightsquigarrow \mathcal{F}_x$ abelsk gruppe

$s \in \mathcal{F}(U) \rightsquigarrow s_x \in \mathcal{F}_x$

When working with sheaves and stalks, it is important to remember the three following properties;

- The germ s_x of s vanishes if and only s vanishes on some neighbourhood of x, *i.e.*, there is an open neighbourhood U of x with $s|_U = 0$.
- All elements of the stalk $\mathcal{F}_x$ are germs, *i.e.*, of the shape s_x for some section s over an open neighbourhood of x.
- The abelian sheaf $\mathcal{F}$ is the zero sheaf if and only if all stalks are zero, *i.e.*, $\mathcal{F}_x = 0$ for all $x \in X$.

Example 1.14 Let $X = \mathbb{C}$, and let $\mathcal{O}_X$ be the sheaf of holomorphic functions in X. If f and g are two sections of $\mathcal{O}_X$ over a neigbourhood U of the point x having the same germ at x, *i.e.*, $f_x = g_x \in \mathcal{O}_{X,x}$, the fact that f and g admit Taylor series expansions around x implies that $f = g$ in the connected component containing x of the set where they both are defined. In fact, the local ring $\mathcal{O}_{X,x}$ is naturally identified with the ring of power series converging in a neighbourhood of x

Morphisms of (pre)sheaves induce maps of stalks

A map $\phi\colon \mathcal{F} \to \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ induces for every point $x \in X$ a map $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ between the stalks. Indeed, one may send a pair (s, U) to the pair $(\phi_U(s), U)$, and since ϕ behaves well with respect to restrictions, this assignment is compatible with the equivalence relations; if (s, U) and (s', U') are equivalent and s and s' coincide on an open set $V \subseteq U \cap U'$, one has

$$\phi_U(s)|_V = \phi_V(s|_V) = \phi_V(s'|_V) = \phi_{U'}(s')|_V.$$

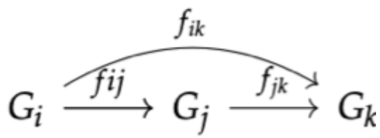
A primer on limits

Another notation for the stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

This is the *direct limit* (also called the *colimit* or the *inductive limit*) of all $\mathcal{F}(U)$ when U runs over the partially ordered set of open sets containing x.

A *directed set* I is a partially ordered set with the property that for each pair of elements $i, j \in I$ there is a third element k such that $i \leqslant k$ and $j \leqslant k$. If I is a directed set and C is a category, a *directed system of objects* in C is a collection $\{G_i\}_{i \in I}$ of objects in C, such that for all $i \leqslant j$ there is a morphism $f_{ij} \colon G_i \to G_j$, and these morphisms satisfy $f_{ii} = id$ and $f_{jk} \circ f_{ij} = f_{ik}$ when $i \leqslant j \leqslant k$.



Direct limits

The direct limit of $\{G_i\}$, denoted by $G = \varinjlim_{i \in I} G_i$, if it exists, is an object in C, equipped with morphisms $g_i : G_i \to G$ which satisfy the following universal property: for any object $H \in \mathsf{C}$ and collection of maps $h_i : G_i \to H$ indexed by I such that $h_i = h_j \circ f_{ij}$ for each $i \leqslant j$, there is a unique map $h : G \to H$ making the following diagram commute for each i:

$$\begin{array}{ccc} G_i & \xrightarrow{h_i} & H \\ {\scriptstyle f_i}\downarrow & \nearrow_{h} & \\ G. & & \end{array}$$

Heuristically, two elements in the direct limit represent the same element in the direct limit if they are 'eventually equal.'

If the G_i are sets (or groups, rings,...), an explicit construction for the direct limit is the quotient $\coprod_{i \in I} G_i / \sim$, where and $g_i \sim g_j$, with $g_i \in G_i$ and $g_j \in G_j$, means that there exists a $k \in I$ with $i \leqslant k$ and $j \leqslant k$ such that $f_{ik}(g) = f_{jk}(h)$.

Example 1.16 Let A be a ring and let $S \subset A$ be a multiplicative subset. Then

$$S^{-1}A = \varinjlim_{s \in S} A_s$$

Example 1.15 In the case I is the set of open neighbourhoods of a point x ordered by inclusion, and $G_U = \mathcal{F}(U)$, we recover the previous definition of the stalk $\mathcal{F}_x$.

Inverse limits

We can similarly define the *inverse limit* (also called the *projective limit* or just the *limit*) of a directed system G_i. The definition just like above, just with the arrows reversed: that is, the maps $G_i \to G_j$ are defined for $j \leqslant i$, and the inverse limit $\varprojlim_{i \in I} G_i$ is an element of C equipped with universal maps *to* each of the G_i, commuting with the maps $G_i \to G_j$.

EXAMPLE 1.17 If all the G_i are subobjects of some fixed object G, and the maps $G_i \leftarrow G_j$ are inclusions $G_j \hookrightarrow G_i$, then

$$\varprojlim_{i \in I} G_i = \bigcap_{i \in I} G_i.$$

☆

1.5 Kernels and images

Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map between two abelian sheaves on X.

Definition 1.18 *The* kernel $\operatorname{Ker}\phi$ *of* ϕ *is a subsheaf of* $\mathcal{F}$ *whose space of sections over* U *is just* $\operatorname{Ker}\phi_U$*, or in other words, the sections in* $\mathcal{F}(U)$ *that map to zero under* $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

The requirement in the definition is compatible with the restriction maps, since $\phi_V(s|_V) = \phi_U(s)|_V$, for any section s over the open set U and any open $V \subseteq U$. Thus we have defined a subpresheaf of $\mathcal{F}$. This is indeed a subsheaf:

Lemma 1.19 *Let* $\phi\colon \mathcal{F} \to \mathcal{G}$ *be a map of abelian sheaves. The kernel* $\operatorname{Ker}\phi$ *is a subsheaf of* $\mathcal{F}$ *having the following two properties.*

i) *Taking the kernel commutes with taking sections:* $\Gamma(U, \operatorname{Ker}\phi) = \operatorname{Ker}\phi_U$*;*
ii) *Forming the kernel commutes with forming stalks:* $(\operatorname{Ker}\phi)_x = \operatorname{Ker}\phi_x$.

One says that the map $\phi : \mathcal{F} \to \mathcal{G}$ is *injective* if $\operatorname{Ker} \phi = 0$. This is, in light of the previous lemma, equivalent to the condition $\operatorname{Ker} \phi_x = 0$ for all x, *i.e.*, that all ϕ_x are injective.

When it comes to images the situation is not as nice as for kernels. One defines the *image presheaf* contained in $\mathcal{G}$ by letting the sections over U be equal to $\operatorname{Im} \phi_U$. However, this is not necessarily a sheaf.

$$(\operatorname{Im} \phi)(U) = \operatorname{Im} \phi_U$$

$$\phi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U).$$

To remedy the situation, we simply make the following definition:

Definition 1.20 *For a morphism $\phi : \mathcal{F} \to \mathcal{G}$ we define the sheaf* $\operatorname{Im}\phi$ *to be the saturation of the image presheaf $U \mapsto \operatorname{Im}\phi_U$, i.e., the smallest subsheaf containing the images.*

Lemma 1.21 *Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of abelian sheaves. The image* $\operatorname{Im}\phi$ *is a subsheaf of $\mathcal{G}$.*

i) For all open subsets U of X one has $\operatorname{Im}\phi_U \subseteq \Gamma(U, \operatorname{Im}\phi)$.

ii) For all $x \in X$ one has $(\operatorname{Im}\phi)_x = \operatorname{Im}\phi_x$.

Proof: i) An element of $t = \phi_U(s)$ of $\operatorname{Im}\phi_X$ is an element of $\mathcal{G}$ which clearly locally lies in $\operatorname{Im}\phi$, so $t \in \Gamma(U, \operatorname{Im}\phi)$.

For ii), let $t_x \in \operatorname{Im}\phi_x$ and pick an $s_x \in \mathcal{F}_x$ with $\phi_x(s_x) = t_x$. We may extend these elements to sections s, t over some open neighbourhood V, so that $\phi_V(s) = t$, and t is a section of $\operatorname{Im}\phi$ over V. This shows that $\operatorname{Im}\phi_x \subseteq (\operatorname{Im}\phi)_x$. Conversely, if t is a section of $\mathcal{G}$ over an open U containing x locally lying in image presheaf, the restriction $t \subseteq V$ lies in $\operatorname{Im}\phi_V$ for some smaller neighbourhood V of x, hence the germ t_x lies in $\operatorname{Im}\phi_x$. ❑

The map $\phi \colon \mathcal{F} \to \mathcal{G}$ is said to be *surjective* if the *image sheaf* $\operatorname{Im} \phi = \mathcal{G}$. This is equivalent to all the stalk-maps ϕ_x being surjective (one says ϕ is surjective on stalks). However, it is important to note that this condition does not imply that the maps ϕ_U are surjective for all U.

Here is a counterexample:

$X = \mathbb{C} - 0$

$\phi : \mathcal{O}_X \to \mathcal{O}_X$ defined by

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U) \qquad f \mapsto f^2$$
$$f \longmapsto f^2$$

$$0 \to K \to \mathcal{O}_X \xrightarrow{f \mapsto f^2} \mathcal{O}_X \to 0$$

⇝ The image presheaf $\operatorname{Im}\phi$ is exactly the presheaf

$$\mathcal{G}(U) = \{ f \in \mathcal{O}(U) \mid f = g^2 \}$$

which is not a sheaf.

We saw that ϕ is surjective locally

(define $\sqrt{z}$ near $z = a \neq 0$ by a power series: $\sqrt{a+w} = \sqrt{a} \cdot \sqrt{1 + w/a}$
$= \sqrt{a} \cdot \sum_{k=0}^{\infty} \binom{1/2}{k} \left(\frac{w}{a}\right)^k$)

but $\phi_X : \mathcal{O}_X(X) \to \mathcal{O}_X(X)$ is not surjective.

1.6 *Exact sequences of sheaves.*

A *complex of sheaves* is a sequence

$$\dots \xrightarrow{\phi_{i-2}} \mathcal{F}_{i-1} \xrightarrow{\phi_{i-1}} \mathcal{F}_i \xrightarrow{\phi_i} \mathcal{F}_{i+1} \xrightarrow{\phi_{i+1}} \dots$$

of maps of abelian sheaves where the composition of any two consecutive maps equals zero, i.e., $\phi_{j-1} \circ \phi_j = 0$ for all j. We say that the sequence is *exact* at $\mathcal{F}_i$ if $\operatorname{Ker} \phi_i = \operatorname{Im} \phi_{i-1}$. The *short exact sequences* are the ones one most frequently encounters. They are sequences of the form

$$0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0 \tag{1.2}$$

that are exact at each stage.

Proposition 1.24 *For a short exact sequence* $0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ *and an open subset* U*, we have the following induced exact sequence*

$$0 \longrightarrow \mathcal{F}'(U) \xrightarrow{\phi_U} \mathcal{F}(U) \xrightarrow{\psi_U} \mathcal{F}''(U). \tag{1.3}$$

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Proof: The map ϕ is injective as a map of sheaves, hence injective on all open sets U, so the sequence above is exact at $\mathcal{F}'(U)$, by Lemma 1.19. To see that it is also exact in the middle, we show that $\operatorname{Ker}(\psi_U) = \operatorname{Im}(\phi_U)$.

It might be helpful to look at the following diagram, for $x \in U$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}'(U) & \xrightarrow{\phi_U} & \mathcal{F}(U) & \xrightarrow{\psi_U} & \mathcal{F}''(U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}'_x & \xrightarrow{\phi_x} & \mathcal{F}_x & \xrightarrow{\psi_x} & \mathcal{F}''_x & \longrightarrow & 0
\end{array}$$

Note that the bottom row is exact, since the sheaf sequence is exact.

That $\operatorname{Im}(\phi_U) \subseteq \operatorname{Ker}(\psi_U)$ is a consequence of taking sections being functorial: since $\psi \circ \phi = 0$, it follows that $\psi_U \circ \phi_U = (\psi \circ \phi)_U = 0$, so everything in $\operatorname{Im} \phi_U$ lies in the kernel of ϕ_U.

Let then see the opposite inclusion $\mathrm{Ker}(\psi_U) \subseteq \mathrm{Im}\,(\phi_U)$. Let $t \in \mathrm{Ker}(\psi_U)$, so that $\psi_U(t) = 0$. Then for all $x \in U$ we have that $\psi_x(t_x) = (\psi_U(t))_x = 0$, so the germ t_x is an element in $\mathrm{Ker}(\psi_x) = \mathrm{Im}\,(\phi_x)$ (where we use exactness at the stalks). That means that for every $x \in U$ there is an element $s'_x \in \mathscr{F}'_x$, say represented by $(s'_{(x)}, V_{(x)})$ for some open neighborhood $V_{(x)} \subseteq U$ of x and $s'_{(x)} \in \mathcal{F}'(V_{(x)})$, such that $\phi_x(s'_x) = t_x$. Then we have that for $x, y \in U$

$$\phi_{V_{(x)} \cap V_{(y)}}(s'_{(x)}|_{V_{(x)} \cap V_{(y)}}) = t|_{V_{(x)} \cap V_{(y)}} = \phi_{V_{(x)} \cap V_{(y)}}(s'_{(y)}|_{V_{(x)} \cap V_{(y)}}),$$

so that by the injectivity of $\phi_{V_{(x)} \cap V_{(y)}}$ (which we have already proved), we get the required condition

$$s'_{(x)}|_{V_{(x)} \cap V_{(y)}} = s'_{(y)}|_{V_{(x)} \cap V_{(y)}}$$

for the gluing of the $s'_{(x)}$ for $x \in U$. Therefore we have a section $s \in \Gamma(U, \mathscr{F})$ with the property that for all $x \in U$

$$s|_{V_{(x)}} = s'_{(x)}.$$

Now we can conclude that for every $x \in U$

$$(\phi_U(s))_x = \phi_x(s_x) = \phi_x(s'_x) = t_x,$$

since $s_x = s'_x$, which gives $\phi_U(s) = t$ as desired. ❑

Let us give a few examples where the surjectivity on the right fails:

1.25 *(Differential operators)* Let $X = \mathbb{C}$ and recall the sheaf $\mathcal{O}_X$ of holomorphic functions and the map $D \colon \mathcal{O}_X \to \mathcal{O}_X$ sending $f(z)$ to the derivative $f'(z)$. There is an exact sequence

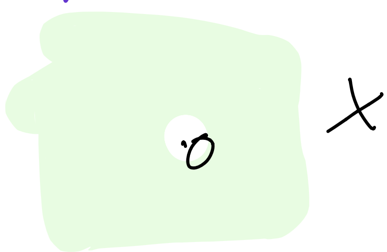
$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \xrightarrow{D} \mathcal{O}_X \longrightarrow 0.$$

lokalt har alle f en antiderivert.

However, taking sections over open sets U we merely obtain the sequence

$$0 \longrightarrow \Gamma(U, \mathbb{C}_X) \longrightarrow \Gamma(U, \mathcal{O}_X) \xrightarrow{D_U} \Gamma(U, \mathcal{O}_X).$$

If U is simply connected, one deduces from Cauchy's integral theorem that every holomorphic function in U is a derivative, so in that case D_U is surjective. On the other hand, if U not simply connected, D_U is not surjective; *e.g.*, if $U = \mathbb{C}\backslash\{0\}$, the function z^{-1} is not a derivative in U. ← har ingen global log z.



1.26 *(The exponential sequence)* Let $X = \mathbb{C} - \{0\}$. The non-vanishing holomorphic functions in an open set $U \subseteq X$ form a *multiplicative* group, and there is a sheaf $\mathscr{O}_X^*$ with these groups as sections. For any f holomorphic in U the exponential $\exp f(z)$ is a section of $\mathscr{O}_X^*$. Hence there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \longrightarrow 0,$$

where the first map sends 1 to $2\pi i$. The rightmost map exp is surjective as a map of sheaves, because non-vanishing functions locally have logarithms. However, over the open set $U = X$, the map is not surjective: the non-vanishing function $f(z) = z$ is not the exponential of a global holomorphic function.

1.7 $\mathscr{B}$-sheaves

Recall that a *basis* for a topology on X is a collection of open subsets $\mathscr{B}$ such that any open set of X can be written as a union of elements of $\mathscr{B}$. In many situations it turns out to be convenient to define a sheaf by saying what it should be on a specific basis for the topology on X.

Let us first make the following definition:

Definition 1.28 *A* $\mathscr{B}$-presheaf $\mathcal{F}$ *consists of the following data:*

i) For each $U \in \mathscr{B}$*, an abelian group* $\mathcal{F}(U)$*;*

ii) For all $U \subseteq V$*, with* $U, V \in \mathscr{B}$*, a restriction map* $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$.

As before, these are required to satisfy the relations $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ *and* $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$*. A* $\mathscr{B}$-sheaf *is a* $\mathscr{B}$*-presheaf satisfying the Locality and Gluing axioms for open sets in* $\mathscr{B}$.

The whole point with the notion of $\mathscr{B}$-sheaved is expressed in the following proposition.

Proposition 1.29 *Let X be a topological space and let $\mathscr{B}$ be a basis for the topology on X. Then*

i) *Every $\mathscr{B}$-sheaf $\mathcal{F}$ extends uniquely to a sheaf on X.*
ii) *If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathscr{B}$-sheaves, then ϕ extends uniquely to a morphism between the corresponding sheaves.*
iii) *The stalk of the extended sheaf at a point x equals the stalk of $\mathcal{F}$ at x; that is, the inductive limit $\varinjlim_{x \in U,\, U \in \mathscr{B}} \mathcal{F}(U)$.*

PROOF: For any open set $U \subseteq X$, we can write U as a union of open sets $U_i \in \mathscr{B}$, and then we can define $\mathcal{F}(U)$ to be the set of elements $s_i \in \prod_i \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j.

1.8 A family of examples – Godement sheaves

Let X be a topological space. Assume that we are given, for each point $x \in X$, an abelian group A_x.

The choice of these groups gives rise to a *sheaf* $\mathscr{A}$ on X whose sections over an open set $U \subseteq X$ are given as

$$\Gamma(U, \mathscr{A}) = \prod_{x \in U} A_x,$$

and whose restriction maps are defined as the natural projections

$$\rho_{UV} \colon \prod_{x \in U} A_x \to \prod_{x \in V} A_x,$$

where $V \subseteq U$ is any pair of open subsets of X.

Definition 1.31 *The sheaf $\mathscr{A}$ is called the* Godement sheaf *of the collection* $\{A_x\}$.

Proposition 1.30 *$\mathscr{A}$ is a sheaf.*

Proof: The Locality condition holds since if the family $\{U_i\}_{i \in I}$ of open sets covers U, any point $x_0 \in U$ lies in some U_{i_0}, so if $s = (a_x)_{x \in U} \in \Gamma(U, \mathscr{A})$ is a section, the component a_{x_0} survives in the projection onto $\Gamma(U_i, \mathscr{A}) = \prod_{x \in U_i} A_x$. Hence if $s|_{U_i} = 0$ for all i, it follows that $s = 0$.

The Gluing condition holds: Assume we are given an open cover $\{U_i\}_{i \in I}$ of U and sections $s_i = (a^i_x)_{x \in U_i} \in \prod_{x \in U_i} A_x$ over U_i matching on the intersections $U_i \cap U_j$. The matching conditions imply that the component of s_i at a point x is the same whatever i is as long as $x \in U_i$. Hence we get a well-defined section s of $\mathscr{A}$ over U by using this common component as the component of s at x. It is clear that $s|_{U_i} = s_i$. ❑

The Godement sheaf associated with a presheaf

Assume $\mathcal{F}$ is a given abelian presheaf on X. The stalks $\mathcal{F}_x$ of $\mathcal{F}$ of course give a collection of abelian groups indexed by points in X, good as any other, and we may form the corresponding Godement sheaf which we denote by $\Pi(\mathcal{F})$.

$$\Pi(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x, \tag{1.4}$$

and the restriction maps are the projections like for any Godement sheaf.

There is an obvious and canonical map

$s \in \mathcal{F}(U) \longrightarrow (s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x$

$$\kappa_{\mathcal{F}} \colon \mathcal{F} \to \Pi(\mathcal{F})$$

sending a section $s \in \mathcal{F}(U)$ to the element $(s_x)_{x \in U}$ of the product in (1.4). This map is functorial in $\mathcal{F}$, for if $\phi \colon \mathcal{F} \to \mathcal{G}$ is a map of sheaves, one has the stalkwise maps $\phi_x \colon \mathcal{F}_x \to \mathcal{G}_x$, and by taking appropriate products of these, we obtain a map $\Pi(\phi) \colon \Pi(\mathcal{F}) \to \Pi(\mathcal{G})$. Over an open set U, we have

$$\Pi(\phi)\big((s_x)_{x \in U}\big) = \big(\phi_x(s_x)\big)_{x \in U},$$

and there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\kappa_{\mathcal{F}}} & \Pi(\mathcal{F}) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \Pi(\phi)} \\ \mathcal{G} & \xrightarrow[\kappa_{\mathcal{G}}]{} & \Pi(\mathcal{G}). \end{array}$$

1.9 *Sheafification*

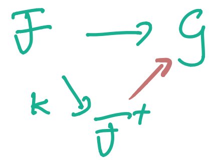
Given any abelian presheaf $\mathcal{F}$ on X, there is a canonical way of defining an abelian sheaf $\mathcal{F}^+$ that in some sense is the sheaf that best approximates it. main properties are summarized in the following

Proposition 1.32 *Given an abelian presheaf $\mathcal{F}$ on X. Then there is a sheaf $\mathcal{F}^+$ and a natural map $\kappa_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ such that:*

i) $\kappa_{\mathcal{F}}$ *is functorial in* $\mathcal{F}$.

$\phi : \mathcal{F} \to \mathcal{G} \rightsquigarrow \phi^+ : \mathcal{F}^+ \to \mathcal{G}^+$

ii) $\kappa_{\mathcal{F}}$ *enjoys the universal property that any map of abelian presheaves* $\mathcal{F} \to \mathcal{G}$ *where* $\mathcal{G}$ *is sheaf, factors through* $\mathcal{F}^+$ *in a unique way. This property characterises* $\mathcal{F}^+$ *up to unique isomorphism.*



iii) If $\mathcal{G}$ *is a sheaf, there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{AbPrSh}_X}(\mathcal{F}, i(\mathcal{G})) = \mathrm{Hom}_{\mathsf{AbSh}_X}(\mathcal{F}^+, \mathcal{G}) \qquad (1.6)$$

where on the right hand side, $i(\mathcal{G})$ *denotes* $\mathcal{G}$ *but considered as a presheaf.*

iv) κ *induces an isomorphism on stalks:* $\mathcal{F}_x \simeq \mathcal{F}^+_x$ *for every* $x \in X$.

Now to the construction:

Recall the canonical map $\kappa \colon \mathcal{F} \to \Pi(\mathcal{F})$ that sends a section s of $\mathcal{F}$ over an open U to the sequence of germs $(s_x)_{x \in U} \in \prod_{x \in U} \mathcal{F}_x = \Gamma(U, \Pi(\mathcal{F}))$. This map certainly kills the 'doomed' sections, *i.e.*, those whose germs all vanish. Now we can get an actual sheaf by taking the image of κ in $\Pi(\mathcal{F})$:

Definition 1.33 *For an abelian presheaf $\mathcal{F}$ on X, we define its* sheafification $\mathcal{F}^+$ *as the image sheaf* $\operatorname{Im} \kappa$ *in* $\Pi(\mathcal{F})$. *In other words, $\mathcal{F}^+$ is the saturation of the subpresheaf $U \mapsto \operatorname{Im} \kappa_U$ in $\Pi(\mathcal{F})$.*

It might help to unravel this definition slightly. Over an open set $U \subseteq X$ the sections of $\mathcal{F}^+$ are given by

$$\mathcal{F}^+(U) = \{(s_x) \in \prod_{x \in U} \mathcal{F}_x | (s_x) \text{ locally lies in } \mathcal{F}\},$$

where, as before, the sentence in the bracket means the following: For each $x \in U$ there exists an open neighbourhood $V \subseteq U$ containing x and a section $t \in \mathcal{F}(V)$ such that for all $y \in V$ we have $s_y = t_y$ in $\mathcal{F}_y$.

Lemma 1.34 *The sheafification $\mathcal{F}^+$ depends functorially on $\mathcal{F}$. Moreover, if $\mathcal{F}$ is a sheaf, $\kappa : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism, so that $\mathcal{F}$ and $\mathcal{F}^+$ are canonically isomorphic.*

Proof: Assume that $\phi\colon \mathcal{F} \to \mathcal{G}$ is a map between two presheaves. Let s be section of $\Pi(\mathcal{F})$ over some open set U, so that s locally lies in $\mathcal{F}$. In other words there is a covering $\{U_i\}$ of U and sections s_i of $\mathcal{F}$ over U_i with $s|_{U_i} = \kappa_{\mathcal{F}}(s_i)$. Hence by (1.5) one has

$$\Pi(\phi)(s|_{U_i}) = \Pi(\phi)(\kappa_{\mathcal{F}}(s_i)) = \kappa_{\mathcal{G}}(\phi(s_i)).$$

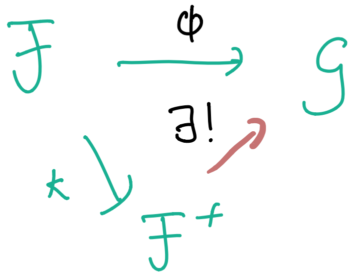
This means that $\Pi(\phi)(s)$ lies locally in $\mathcal{G}$, and $\Pi(\phi)$ takes $\mathcal{F}^+$ into $\mathcal{G}^+$. Moreover, there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \xrightarrow{\kappa_{\mathcal{F}}} & \mathcal{F}^+ & \longrightarrow & \Pi(\mathcal{F}) \\ \phi\downarrow & & \phi^+\downarrow & & \downarrow\Pi(\phi) \\ \mathcal{G} & \xrightarrow{\kappa_{\mathcal{G}}} & \mathcal{G}^+ & \longrightarrow & \Pi(\mathcal{G}) \end{array}$$

In case $\mathcal{F}$ is a sheaf, the map $\kappa_{\mathcal{F}}$ maps $\mathcal{F}$ injectively into $\Pi(\mathcal{F})$ and $\mathcal{F} = \operatorname{Im} \kappa_{\mathcal{F}}$ is its own saturation, hence $\kappa_{\mathcal{F}}$ is an isomorphism. ❑

Lemma 1.35 *Given an abelian presheaf $\mathcal{F}$ on X. Then the sheaf $\mathcal{F}^+$ and the natural map $\kappa : \mathcal{F} \to \mathcal{F}^+$ enjoys the universal property that any map of abelian presheaves $\mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is sheaf, factors through $\mathcal{F}^+$ in a unique way. This property characterises $\mathcal{F}^+$ up to unique isomorphism.*

Proof: If $\mathcal{G}$ in the diagram above is a sheaf, the map $\kappa_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}^+$ is an isomorphism and $\phi^+ \circ \kappa_G^{-1}$ provides the wanted factorization. The uniqueness statement follows formally: Given two abelian sheaves $\mathcal{F}^+$ and $\mathcal{F}'$ satisfying the above, we get by the universal properties two maps $\mathcal{F}^+ \to \mathcal{F}'$ and $\mathcal{F}' \to \mathcal{F}^+$, whose compositions are the identity by uniqueness. ❑



Lemma 1.36 *Sheafification preserves stalks:* $\mathcal{F}_x = (\mathcal{F}^+)_x$ *via* κ_x.

Proof: The map $\kappa_x : \mathcal{F}_x \to (\mathcal{F}^+)_x$ is injective, because $\mathcal{F}_x \to (\Pi(\mathcal{F}))_x$ is injective. To show that it is surjective, suppose that $\bar{s} \in (\mathcal{F}^+)_x$. We can find an open neighbourhood U of x such that $\bar{s}$ is the equivalence class of (s, U) with $s \in \mathcal{F}^+(U)$. By definition, this means there exists an open neighbourhood $V \subseteq U$ of x and a section $t \in \mathcal{F}(V)$ such that $s|_V$ is the image of t in $\Pi(\mathcal{F})(V)$. Clearly the class of (t, V) defines an element of $\mathcal{F}_x$ mapping to $\bar{s}$. ❑

A group $\rightsquigarrow$ $A'_X(U) = A \quad \forall U \subseteq X$

Examples

1.37 *(Constant sheaves)* Recall Example 1.8 in which we showed that the constant presheaf given by $A'_X(U) = A$ is usually not a sheaf (where A is an abelian group). In this case, the sheafification is exactly the sheaf A_X defined by

$$\Gamma(U, A_X) = \prod_{\pi_0(U)} A, = \{ f : U \to A \text{ continuous} \}$$

where $\pi_0(U)$ denotes the set of connected components of the open set U.

Define $\varphi : A_X \to (A'_X)^+$ by $\varphi_U : A_X(U) \longrightarrow \prod_{x \in U} A$

$(f : U \to A) \mapsto (f(x))_{x \in U}$

f locally constant $\Rightarrow$ $\varphi_U(f)$ locally lies in A'_X

$\Rightarrow$ φ takes values in $(A'_X)^+$.

On stalks: $(A_X)_x = A$, $(A'_X)^+_x = A$ and $\varphi_x : A \longrightarrow A$ is the identity

$\Rightarrow$ φ isomorphism.

Why we need to sheafify things

Let X be an affine variety

$A(X)$ = coordinate ring of X

$k(X)$ = field of rational functions

Consider the "naive sheaf of regular functions"

$$\mathcal{O}'_X(U) = \left\{ f : U \to k \;\middle|\; f = \frac{g}{h} \text{ where } g, h \in A(X) \right\}$$

This is not a sheaf:

$xy - zw = 0$

$\Rightarrow \frac{z}{x} = \frac{y}{w}$

$X = Z(xy - zw) \subset \mathbb{A}^4$

$U = D(x) \quad \rightsquigarrow \quad \frac{z}{x}$ gives an element of $\mathcal{O}'_X(U)$

$V = D(w) \quad \rightsquigarrow \quad \frac{y}{w}$ gives an element of $\mathcal{O}'_X(V)$

Since $xy - zw = 0$, these agree on the overlap.

However, there is no $\frac{g}{h} \in \mathcal{O}'_X(U \cup V)$ s.t

$$\left.\frac{g}{h}\right|_U = \frac{z}{x} \quad \text{and} \quad \left.\frac{g}{h}\right|_V = \frac{y}{w}$$

This is related to the fact that $\dfrac{k[x,y,z,w]}{(xy - zw)}$ is not a UFD.

However, the two sections <u>do</u> glue to a regular function $f : U \cup V \to k$, that is, a section of

$$\mathcal{O}_X(W) = \left\{ f : W \to k \;\middle|\; \begin{array}{l} \text{for every } p \in W \;\; \exists\, g, h \in A(X) \\ \text{s.t } f = g/h \text{ in a nbh} \\ \text{of } p \end{array} \right\}$$

$\Rightarrow$ the structure sheaf = sheafification of the presheaf $\mathcal{O}_X'$.

Quotient sheaves

Sheafification $\Longrightarrow$ can create quotient sheaves.

$\mathcal{G} \subseteq \mathcal{F}$ subsheaf $\rightsquigarrow$ presheaf $Q(U) = \mathcal{F}(U) / \mathcal{G}(U)$

$\mathcal{F}/\mathcal{G} :=$ sheafification of Q

Cokernel

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, we define

$\operatorname{coker} \phi :=$ sheafification of the presheaf
$U \mapsto \operatorname{coker}(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$

$\rightsquigarrow$ exact sequence

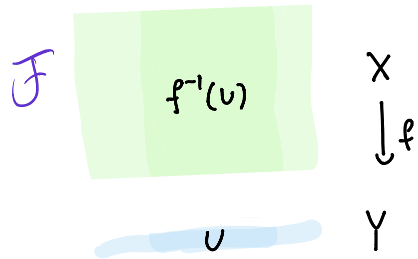
$$0 \to \ker \phi \to \mathcal{F} \to \mathcal{G} \to \operatorname{coker} \phi \to 0$$

(exact because exact at stalks)

Example 1.39 To see why we have to sheafify in these constructions, consider again the exponential map from Example 1.26. The naive presheaf $U \mapsto \operatorname{Coker} \exp(U)$ is not a sheaf: The class of the function $f(z) = z$ restricts to 0 in $\operatorname{Coker} \exp$ on sufficiently small open sets, but it is itself not zero (since otherwise we would be able to define a global logarithm on $\mathbb{C} - 0$). ☆

$$\therefore \quad (\operatorname{coker} \exp)^+ = 0 .$$

1.11 *The pushforward of a sheaf*



Let X and Y be two topological spaces with a continuous map $f\colon X \to Y$ between them. Assume that $\mathcal{F}$ is an abelian sheaf on X. This allows us to define an abelian sheaf $f_*\mathcal{F}$ on Y by specifying the sections of f_*F over the open set $U \subseteq Y$ to be

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U),$$

and letting the restriction maps $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ be the ones from $\mathcal{F}$.

Definition 1.40 *The sheaf $f_*\mathcal{F}$ is called the* pushforward sheaf *or* the direct image *of $\mathcal{F}$.*

Lokalitet: $s \in f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$ $\quad s|_{U_i} = 0$

$s|_{U_i} \in \mathcal{F}(f^{-1}U_i)$

$\Rightarrow s = 0$ fordi $f^{-1}U_i$ overdekning av $f^{-1}U$.

<u>Liming:</u> $s_i \in f_*\mathcal{F}(U_i)$ s.a. $s_i = s_j$ på $U_i \cap U_j$

$\mathcal{F}$ knippe $\qquad \mathcal{F}(f^{-1}U_i)$ (= $f_*\mathcal{F}(U_i)$) $\qquad s_i = s_j$ i $\mathcal{F}(f^{-1}U_i \cap f^{-1}U_j)$

$\Rightarrow$ s_i liver til et element i $\mathcal{F}(f^{-1}U) = f_*\mathcal{F}(U)$. □

Example 1.42 Consider an affine variety $X \subseteq \mathbb{A}^n$ and let $i : X \to \mathbb{A}^n$ be the inclusion. For each open $U \subseteq \mathbb{A}^n$ define

$$\mathcal{I}_X(U) = \{f \in \mathcal{O}_{\mathbb{A}^N}(U) | f(x) = 0 \,\forall\, x \in X\}.$$

Then $\mathcal{I}_X$ is a sheaf (of ideals) and we have an exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{A}^n} \to i_* \mathcal{O}_X \to 0$$

☆

Merk: $f : X \longrightarrow *$ $\Rightarrow$ $f_* \mathcal{F} = \Gamma(X, \mathcal{F})$

LEMMA 1.43 *The functor f_* is left exact. That is, given an exact sequence of sheaves on X*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

then the following sequence is exact

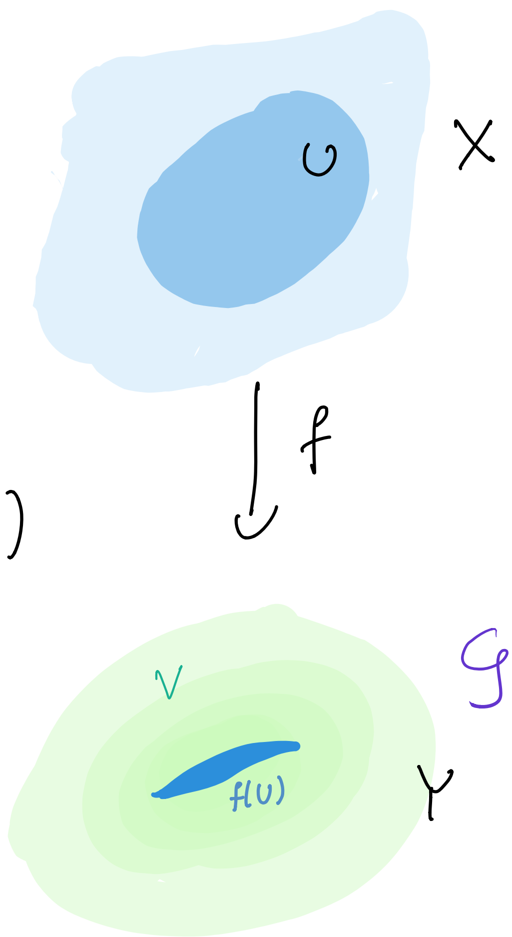
$$0 \longrightarrow f_*\mathcal{F}' \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{F}''.$$

Inverse image sheaf

Given a continuous map $f: X \to Y$

$\mathcal{G}$ sheaf on Y

$\rightsquigarrow$ inverse image sheaf $f^{-1}\mathcal{G}$



Construction:

For $U \subseteq X$ open, let

$$f_p^{-1}\mathcal{G}(U) = \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$$

$f(U)$ need not be open, so we use open sets containing it

$\rightsquigarrow$ $f_p^{-1}\mathcal{G}$ is a presheaf on X.

Defn We define the inverse image sheaf $f^{-1}\mathcal{G}$ as the sheafification of $f_p^{-1}\mathcal{G}$.

Note in particular that the stalk of $f^{-1}\mathcal{G}$ at a point $x \in X$ is isomorphic to $\mathcal{G}_{f(x)}$. Indeed, it suffices to verify this on the level of presheaves:

$$\begin{aligned}
(f_p^{-1}\mathcal{G})_x &= \varinjlim_{U \ni x} f_p^{-1}\mathcal{G}(U) = \varinjlim_{U \ni x} \varinjlim_{V \supseteq f(U)} \mathcal{G}(V) \\
&= \varinjlim_{V \ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}
\end{aligned}$$

The adjoint property of f^{-1} and f_*

THEOREM 1.48 *Let $f : X \to Y$ be a morphism, let $\mathcal{F}$ be an abelian sheaf on X and let $\mathcal{G}$ be an abelian presheaf on Y. Then we have a natural bijection*

$$\mathrm{Hom}_{\mathsf{AbPrSh}_Y}(\mathcal{G}, f_*\mathcal{F}) \simeq \mathrm{Hom}_{\mathsf{AbSh}_X}(f^{-1}\mathcal{G}, \mathcal{F})$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

$f^{-1}\mathcal{G}$ satisfies the natural universal property:

morphisms $\mathcal{G} \to f_*\mathcal{F}$ $\Longleftrightarrow$ morphisms $f^{-1}\mathcal{G} \to \mathcal{F}$

proof: See the notes!