Sheaves of $\mathcal{O}_X$-modules

an $\mathcal{O}_X$-*module structure* on an abelian sheaf $\mathcal{F}$ is defined as a family of multiplication maps $\mathcal{F}(U) \times \mathcal{O}_X(U) \to \mathcal{F}(U)$—one for each open subset U of X—making the space of sections $\mathcal{F}(U)$ into a $\mathcal{O}_X(U)$-module in a way compatible with all restrictions. That is, for every pair of open subsets $V \subset U$,

$$\begin{array}{ccc} \mathcal{F}(U) \times \mathcal{O}_X(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) \times \mathcal{O}_X(V) & \longrightarrow & \mathcal{F}(V). \end{array}$$

Maps, or homomorphisms, of $\mathcal{O}_X$-modules are simply maps $\alpha\colon \mathcal{F} \to \mathcal{G}$ between $\mathcal{O}_X$-modules considered as abelian sheaves, respecting the multiplication by sections of $\mathcal{O}_X$. That is, for any open U the map $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a $\mathcal{O}_X(U)$-module homomorphism.

$\rightsquigarrow$ Mod_X = kategorien av $\mathcal{O}_X$-moduler

Mod_X er en additiv kategori: ∘ $\bigoplus \mathcal{F}_i$

∘ $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \{ \alpha : \mathcal{F} \to \mathcal{G} \}$ $\mathcal{O}_X$-modul

∘ ker, coker, im

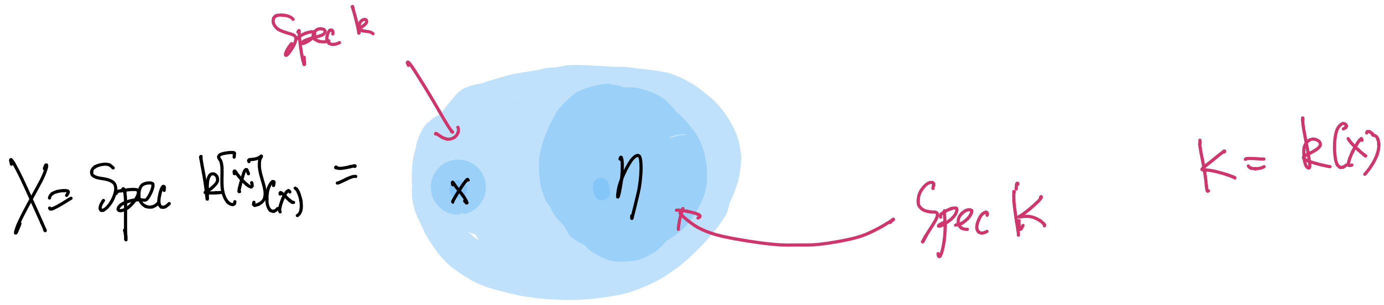
For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ we also define the *tensor product*, denoted by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$. As in many other cases, the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is defined by first describing a presheaf that subsequently is sheafified. The sections of the presheaf, temporarily denoted by $\mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G}$, are defined in the natural way by

$$(\mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U). \tag{10.1}$$

↑ (skal se eksempler på at dette ikke er et knippe (selv på $\mathbb{P}^1$))

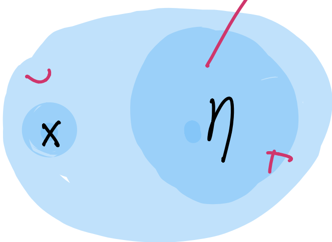
$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

There is also a *sheaf of $\mathcal{O}_X$-homomorphisms* between $\mathcal{F}$ and $\mathcal{G}$. Recall the sheaf $\mathscr{H}om(\mathcal{F},\mathcal{G})$ of homomorphisms between the abelian sheaves $\mathcal{F}$ and $\mathcal{G}$ whose sections over an open set U is the group $\mathrm{Hom}(\mathcal{F}|_U,\mathcal{G}|_U)$ of homomorphisms between the restrictions $\mathcal{F}|_U$ and $\mathcal{G}|_U$. Inside this group one has the subgroup of the maps being $\mathcal{O}_X$-homomorphisms, and these subgroups, for different open sets U, are respected by the restriction map. So they form the sections of a presheaf, that turns out to be a sheaf, and that is the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ of $\mathcal{O}_X$-homomorphisms from $\mathcal{F}$ to $\mathcal{G}$.



EXAMPLE 10.1 *(Modules on spectra of DVR's)* Modules on the prime spectrum of a discrete valuation ring R are particularly easy to describe.

Recall that the scheme $X = \operatorname{Spec} R$ has only two non-empty open sets, the whole space X itself and the singleton $\{\eta\}$ where η denotes the generic point. The singleton $\{\eta\}$ is the underlying set of the open subscheme $\operatorname{Spec} K$, where K denotes the fraction field of R.



$R = k[x]_{(x)}$

We claim that to give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space N and a R-module homomorphism $\rho : M \to N$.

Indeed, given an $\mathcal{O}_X$-module $\mathcal{F}$, we get an R-module $M = \mathcal{F}(X)$, and a vector space $N = \mathcal{F}(\{\eta\})$ over K. The homomorphism ρ is just the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{\eta\})$. Conversely, given the data M, N, ρ, we can define $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = N$. The map $\rho : \mathcal{F}(X) \to \mathcal{F}(\{\eta\})$ makes $\mathcal{F}$ into an $\mathcal{O}_X$-module.

ex $M = R$, $\rho = 0$ $\rightsquigarrow$ ok.

Note that the restriction map can be just any R-module homomorphism $M \to N$. In particular, it can be the zero homomorphism, and in that case M and N can be completely arbitrary modules. Again, this illustrates the versatility of general $\mathcal{O}_X$-modules. ☆

ex : Godement knipper

Recall the Godement construction from Chapter 1. Given any collection of abelian groups $\{A_x\}_{x \in X}$ indexed by the points x of X. We defined a sheaf $\mathcal{A}$ whose sections over an open subset U was $\prod_{x \in U} A_x$, and whose restriction maps to smaller open subsets were just the projections onto the corresponding smaller products. Requiring that each A_x be a module over the stalk $\mathcal{O}_{X,x}$ makes $\mathcal{A}$ into an $\mathcal{O}_X$-module; indeed, the space of sections $\Gamma(U, \mathcal{A}) = \prod_{x \in U} A_x$ is automatically an $\mathcal{O}_X(U)$-module, the multiplication being defined componentwise with the help of the stalk maps $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$. Clearly this module structures is compatible with the projections, and thus makes $\mathcal{A}$ into an $\mathcal{O}_X$-module. ☆

$$\mathrm{Supp}(M) = \{ \mathfrak{p} \mid M_{\mathfrak{p}} \neq 0 \}$$

$$0 \to M' \to M \to M'' \to 0$$

$$\mathrm{Supp}\, M = \mathrm{Supp}\, M' \cup \mathrm{Supp}\, M''$$

The support of a sheaf

Definition 10.3 *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The* support of $\mathcal{F}$, *$Supp(\mathcal{F})$, is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$. For $s \in \mathcal{F}(X)$ we define the* support of s *as the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero. We denote this by $Supp(s)$.*

ex

$X = \mathrm{Spec}\, k[x,y]$

$$M = \frac{k[x,y]}{(y^2 - x^3)}$$

$\mathrm{Supp}(M) = V(y^2 - x^3)$

Note that if $s \in \mathcal{F}(X)$ is a section, and x is a point such that $s_x = 0$ in $\mathcal{F}_x$, then there is an open neighbourhood $V \subset X$ containing x such that $s_y = 0$ for all $y \in V$. It follows that the support of s is a closed subset of X.

In contrast, the support of a sheaf of modules is in general not closed. Indeed, as before, we can get strange sheaves by taking any non-closed subset Z of your favourite ringed space, and define a Godement sheaf $\mathscr{A}$ with the property that $\mathscr{A}_x \neq 0$ if and only if $x \in Z$.

10.2 *Pushforward and Pullback of $\mathcal{O}_X$-modules*

In Chapter **??** we introduced two functors between the categories AbSh_X and AbSh_Y associated with a continuous map $f\colon X \to Y$ between topological spaces; the *pushforward* functor f_* and the *inverse image* functor f^{-1}. In this section we parallel these two constructions when f is a morphism of schemes to obtain functors f_* and f^* between Mod_X and Mod_Y. They form an adjoint pair of functors.

$f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}U)$

Pushforward

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. If $\mathcal{F}$ is an abelian sheaf on X recall that the pushforward $f_*\mathcal{F}$ may be considered the restriction of $\mathcal{F}$ to the subcategory of open_X consisting of inverse images of opens in Y. When $\mathcal{F}$ is an $\mathcal{O}_X$-module, it is then clear that $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-modules in a natural way, and hence, via the map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$, it has a natural $\mathcal{O}_Y$-module structure.

Definition 10.4 *The above $\mathcal{O}_Y$-module $f_*\mathcal{F}$ is called the* direct image of $\mathcal{F}$ *under f.*

This construction is clearly functorial in $\mathcal{F}$, and so we obtain a functor $f_* : \mathsf{Mod}_X \to \mathsf{Mod}_Y$. The pushforward f_* is a left exact functor, which follows easily from Lemma **??** in Chapter **??**. It is also functorial in f in the sense that $(f \circ g)_* = f_* \circ g_*$ when f and g are composable morphism of schemes; indeed, this follows easily from $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ and $(f \circ g)^\# = g^\# \circ f^\#$.

$f : X \to Y \qquad \mathcal{G} \in \mathrm{Mod}_Y$

$f^{-1}\mathcal{G} \ : \qquad f^{-1}\mathcal{G}(U) = \left(\varinjlim_{V \supset f(U)} \mathcal{G}(V) \right)^{+}$

$f^{*}\mathcal{G}$

Pullback

The pullback of a sheaf of $\mathcal{O}_Y$-modules is a little bit more difficult to define. Recall that we in Chapter **??** defined (Definition **??**) the inverse image $f^{-1}\mathcal{G}$ of a sheaf $\mathcal{G}$ by sheafifying the presheaf given by assigning to an open subset $U \subset X$

$$\mathrm{Hom}(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathrm{Hom}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$$

$$\mathcal{O}_Y \xrightarrow{f^\#} f_*\mathcal{O}_X$$

the direct limit of all $\mathcal{G}(V)$ where V contains $f(U)$. When $\mathcal{G}$ is a $\mathcal{O}_Y$-module, this sheaf is naturally a $f^{-1}\mathcal{O}_Y$-module. We can make it into an $\mathcal{O}_X$-module using the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (which makes $\mathcal{O}_X$ an $f^{-1}\mathcal{O}_Y$-algebra), and taking the tensor product:

$$f^*\mathcal{G} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{G}$$

$$\mathcal{G} = \mathcal{O}_Y \qquad \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_Y$$

The association $\mathcal{G} \mapsto f^*\mathcal{G}$ is functorial, so we get a functor $f^* : \mathsf{Mod}_{\mathcal{O}_Y} \to \mathsf{Mod}_{\mathcal{O}_X}$. The above $\mathcal{O}_X$-module is called the *pullback of* $\mathcal{G}$ under f. Note in particular that $f^*\mathcal{O}_Y = \mathcal{O}_X$, as $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$.

Proposition 10.5 *For a point $x \in X$ we have the following expression for the stalk*

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

Proof: This follows from the facts that taking stalks commutes with sheafification and tensor products, and $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$. ❑

Adjoint properties of f_*, f^*

At first sight, the definition of the pullback might seem a bit out of the blue. It is defined from $f^{-1}\mathcal{G}$, tensoring with $\mathcal{O}_X$ over $f^{-1}\mathcal{O}_Y$ to rig it into being a $\mathcal{O}_X$-module. However, as like in the case of the inverse image functor f^{-1}, the important point is what the sheaf does, rather than how it is explicitly defined. In the present case, the pullback is the adjoint of a functor which is easy to understand, namely f_*:

Proposition 10.6 *The functors f_*, f^* between $\mathsf{Mod}_{\mathcal{O}_X}, \mathsf{Mod}_{\mathcal{O}_Y}$ are adjoint. In other words, if $\mathcal{F} \in \mathsf{Mod}_{\mathcal{O}_X}, \mathcal{G} \in \mathsf{Mod}_{\mathcal{O}_Y}$, there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

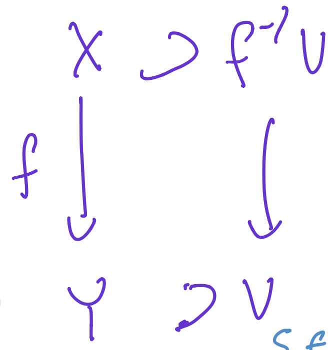
Proof: See Exercise **??**. ❑

In particular, we the maps $\mathrm{id}_{f^*\mathcal{G}}$ and $\mathrm{id}_{f_*\mathcal{F}}$ provide us with the canonical maps

$$\eta : \mathcal{G} \to f_* f^* \mathcal{G}, \qquad \nu : f^* f_* \mathcal{F} \to \mathcal{F}$$

We already saw previously that f_* is a left-exact functor. This implies that f^* is right-exact, by general properties of adjoint functors.

$$\eta : \mathcal{G} \to f_* f^* \mathcal{G},$$



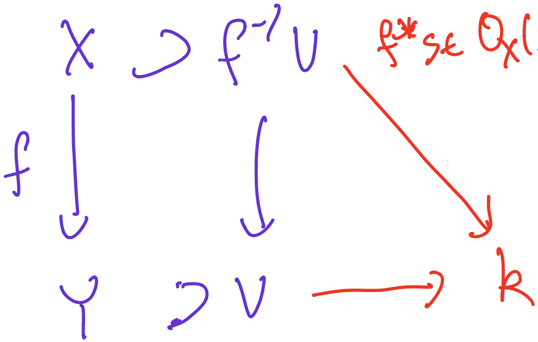
$\eta(V) : \mathcal{G}(V) \longrightarrow \Gamma(V, f_* f^* \mathcal{G})$

$= \Gamma(f^{-1}V, f^*\mathcal{G})$

$s \in \mathcal{G}(V)$

Pullback of sections

We can also pull back sections of $\mathcal{G}$. If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, and $s \in \mathcal{G}(V)$, then we get a section $f^*(s) = \eta(s) \in \Gamma(f^{-1}(V), f^*\mathcal{G})$ by the map $\eta : \mathcal{G} \to f_* f^* \mathcal{G}$.



$f^* s \in \mathcal{O}_X(f^{-1}V)$

$s \in \mathcal{O}_Y(V)$

analogt til f^* fra alg. geo I.

$f^* \mathcal{O}_Y = \mathcal{O}_X$

10.3 *Quasi-coherent sheaves*

M A-modul $\rightsquigarrow$ $\widetilde{M}$ knippe av $\mathcal{O}_X$-moduler

$M \mapsto \widetilde{M}$ "tilde-funktoren"

Quasi-coherent sheaves on affine schemes

In this section we work over an affine scheme $X = \operatorname{Spec} A$. For each A-module M we shall define an $\mathcal{O}_X$-module $\widetilde{M}$, the construction of which completely parallels what we did when constructing the structure sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} A$.

$D(f) \rightsquigarrow \mathcal{O}_X(D(f)) = A_f$

Letting $\mathscr{B}$ again be the base of the topology consisting of distinguished open subsets on X, we define a $\mathscr{B}$-presheaf $\widetilde{M}$ by letting sections be given by

$$(\widetilde{M})(D(f)) = M_f.$$

and the restriction maps are the canonical localization maps: when $D(g) \subset D(f)$, there is a canonical localization map $M_f \to M_g$; it sends mf^{-n} to $a^n mg^{-nm}$ where $g^n = af$. The same proof as for $\mathcal{O}_X$ (Proposition **??**) shows that this is actually a $\mathscr{B}$-sheaf, and hence gives rise to a unique *sheaf* $\widetilde{M}$ on X.

Defn

We will say that a sheaf $\mathcal{F}$ on $X = \operatorname{Spec} A$ is *quasi-coherent* if is isomorphic to a sheaf of the form $\widetilde{M}$ for some A-module M

$D(g) \subset D(f)$

$$\begin{array}{ccc} M_f & \longrightarrow & N_f \\ \downarrow & & \downarrow \\ M_g & \longrightarrow & N_g \end{array} \quad \rightsquigarrow \quad \widetilde{M} \to \widetilde{N}$$

The tilde-construction is functorial in M. For any A-module homomorphism $\alpha : M \to N$ there is an obvious way of obtaining an $\mathcal{O}_X$-module homomorphism $\tilde{\alpha} : M \to N$; indeed, the maps $\alpha_f : M_f \to N_f$ are $\mathcal{O}_X(D(f))$-linear homomorphisms compatible with localization maps, and thus induce a map between $\widetilde{M}$ and $\tilde{\mathrm{N}}$. Clearly one has $\widetilde{\phi \circ \psi} = \tilde{\phi} \circ \tilde{\psi}$, and the "tilde-operation" is therefore a functor $\mathsf{Mod}_A \to \mathsf{Mod}_{\mathcal{O}_X}$.

Proposition 10.8 *Let A be a ring and M and A-module. The sheaf $\widetilde{M}$ on* Spec A *has the following three properties.*

i) *Stalks: let $x \in \operatorname{Spec} A$ be a point whose corresponding prime ideal is $\mathfrak{p}$, the stalk $\widetilde{M}_x$ of M at $x \in X$ is $\widetilde{M}_x = M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$;*

ii) *Sections over distinguished open sets: if $f \in A$, one has $\Gamma(D(f), \widetilde{M}) = M_f = M \otimes_A A_f$; in particular it holds true that $\Gamma(X, \widetilde{M}) = M$;* $f = 1$

iii) *Sections over arbitrary open sets: for any open subset U of* Spec A *covered by the distinguished sets $\{D(f_i)\}_{i \in I}$, there is an exact sequence*

$$0 \longrightarrow \Gamma(U, \widetilde{M}) \xrightarrow{\alpha} \prod_i M_{f_i} \xrightarrow{\rho} \prod_{i,j} M_{f_i f_j}$$

The tildes enjoy a certain universal property among the $\mathcal{O}_X$-modules on $X = \operatorname{Spec} A$. Assume that an $\mathcal{O}_X$-module $\mathcal{F}$ is given on X, and let $M = \mathcal{F}(X)$ denote the global sections of $\mathcal{F}$. There is a natural map $\beta : \widetilde{M} \to \mathcal{F}$ of $\mathcal{O}_X$-modules.

$$\widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F}$$

$\mathcal{O}_X$-modul homomorfi

blir en isomorfi dersom $\mathcal{F}$ er kvasikoherent

$M = \mathcal{F}(X) \ni m$ $\qquad$ $\widetilde{M}(D(f)) = M_f \ni mf^{-n} \rightsquigarrow f^{-n} \cdot m|_{D(f)}$ $\qquad$ $A_f \downarrow$ $\quad$ $\Gamma(D(f), \mathcal{F}) \downarrow$

As usual, it suffices to tell what the map does to sections over the distinguished opens. The sheaf $\mathcal{F}$ being an $\mathcal{O}_X$-module, multiplication by f^{-1} in the space of sections $\Gamma(D(f), \mathcal{F})$ makes sense since $\Gamma(D(f), \mathcal{O}_X) = A_f$. Hence we may send the section $mf^{-n} \in M_f$ of $\widetilde{M}$ to the section of $\mathcal{F}$ over $D(f)$ obtained by multiplying the restriction of m to $D(f)$ by f^{-n}; *i.e.* we send mf^{-n} to $f^{-n} \cdot m|_{D(f)}$. For later reference we state this observation as a lemma:

Lemma 10.9 *Given an $\mathcal{O}_X$-module $\mathcal{F}$ on the affine scheme $X = \operatorname{Spec} A$. Then there is a* unique *$\mathcal{O}_X$-module homomorphism*

$$\beta\colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$$

inducing the identity on the spaces of global sections. Moreover, it is natural in the sense that if $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_X$-module inducing the map $a\colon \mathcal{F}(X) \to \mathcal{G}(X)$ on global sections, one has $\beta_{\mathcal{G}} \circ \widetilde{a} = \alpha \circ \beta_{\mathcal{F}}$.

Lemma 10.10 *In the canonical identification of the distinguished open subset $D(f)$ with* $\operatorname{Spec} A_f$*, the* $\mathcal{O}_X$*-module* $\widetilde{M}$ *restricts to* $\widetilde{M_f}$*.*

Proof: As $\Gamma(D(f), \widetilde{M}) = M_f$, there is a map $\beta_f : \widetilde{M_f} \to \widetilde{M}|_{D(f)}$ that on distinguished open subsets $D(g) \subset D(f)$ induces an isomorphism between the two spaces of sections, both being equal to the localization M_g. ❑

Egenskaper til ~-funktoren:

$\mathcal{F}, \mathcal{G}$ $\quad (\mathcal{F} \otimes \mathcal{G})' = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

prekнippe ⟶ knippifiser

PROPOSITION 10.12 *Assume that A is a ring and let $X = \operatorname{Spec} A$. The functor from the category Mod_A of A-modules to the category $\mathsf{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules given by $M \to \widetilde{M}$ enjoys the following three properties*

i) *It is a fully faithful additive and exact functor.*

ii) *There is a canonical isomorphism $\widetilde{M \otimes_A N} \simeq \widetilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$.*

iii) *If M is of finite presentation, then there is a canonical isomorphism of sheaves $\operatorname{Hom}_A(M, N)^{\sim} \simeq \mathscr{H}om_{\mathcal{O}_X}(\widetilde{M}, \tilde{N})$.*

åpen U

$$\operatorname{Hom}_{\mathcal{O}_U}(\tilde{M}|_U, \tilde{N}|_U)$$

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

Endelig presentasjon:

$M = k[x,y]/(x^2, xy, y^3)$ $\quad A = k[x,y]$

$$k[x,y]^3 \longrightarrow k[x,y] \longrightarrow M \longrightarrow 0$$

$e_1 \longrightarrow x^2$

$e_2 \longrightarrow xy$

$e_3 \longrightarrow y^3$

Lemma 10.11 *For any two A-modules M and N, the association* $\phi \to \tilde{\phi}$ *gives a bijection* $\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \tilde{N})$ *whose inverse is* $\alpha \mapsto \alpha(X)$,

Proof: That $\tilde{\phi} = \alpha$ when $\phi = \alpha(X)$ may be checked on distinguished open sets where it boils down to the definition of $\tilde{\phi}$ and the fact that α commutes with the localisation maps. That $\Gamma(X, \tilde{\phi}) = \phi$ follows directly from the definition of $\tilde{\phi}$. ❑

$\rightsquigarrow$ $\sim$ er full + trofast.

Given an exact sequence of A-modules:

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

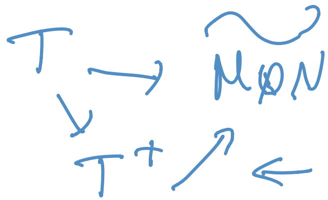
That the induced sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow 0$$

is exact is a direct consequence of the three following facts. The stalk of a tilde-module $\widetilde{M}$ at the point x with corresponding prime ideal $\mathfrak{p}$ is $M_\mathfrak{p}$, localization is an exact functor, and finally, a sequence of abelian sheaves is exact if and only if the sequence of stalks at every point is exact.

$$0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$$

$$\widetilde{M \otimes_A N} = \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = T^+$$



For the tensor product, let T denote the presheaf $U \mapsto \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \tilde{N}(U)$. We have the map of presheaves $T \to \widetilde{M \otimes_A N}$ (on $U = D(f)$ it is the isomorphism $M_f \otimes_{A_f} N_f \simeq (M \otimes_A N)_f$; it sends $m/f^a \otimes n/f^b$ to $(m \otimes n)/f^{a+b}$). After sheafifying, we get a the desired map of sheaves

$$\widetilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to \widetilde{M \otimes_A N}.$$

T^+

This is an isomorphism, since it is an isomorphism over every distinguished open set.

$$\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

$$\widetilde{M} \rightsquigarrow f_* \widetilde{M}$$

B-modul

Functoriality

Pushforward

Suppose we are given a morphism $f: X \to Y$ between the two affine schemes $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We let $\phi = f^\sharp(X): A \to B$ be the ring map corresponding to f. If M is an B-module, can one describe the sheaf $f_* \widetilde{M}$ on Y?

$$M$$

$$A \xrightarrow{\phi} B \qquad \rightsquigarrow \qquad M_A = M \text{ mit } A\text{-modulstruktur}$$

$$a \cdot m := \phi(a) \cdot m$$

Proposition 10.14 $f_*\widetilde{M} = \widetilde{M_A}$.

Proof: Let $a \in A$. The crucial observation is that $f^{-1}(D(a)) = D(\phi(a))$. (Indeed, a prime ideal $\mathfrak{p} \subset A$ satisfies $a \in \phi^{-1}(\mathfrak{p})$ if and only if $\phi(a) \in \mathfrak{p}$.) Then note that

$$\Gamma(D(a), f_*(\widetilde{M})) = \Gamma(D(\phi(a)), \widetilde{M}) = M_{\phi(a)}$$

Note that a acts on M_A as multiplication by $\phi(a)$: This means that the module on the right is isomorphic to $(M_A)_a = \Gamma(D(a), \widetilde{M_A})$. Thus there is an isomorphism of $\mathscr{B}$-sheaves $f_*\widetilde{M} \simeq \widetilde{M_A}$, and we are done. ❑

Pullback

Recall the notion of pullback of a sheaf via a morphism $f : X \to Y$. This is a relatively complicated operation, since it involves taking a direct limit, a tensor product, and finally a sheafification. The next result tells us that for $\mathcal{G}$ a sheaf of the form $\widetilde{M}$ on Y, we have a much simpler description of the pullback $f^*\mathcal{G}$, which will allow us to do local computations more easily.

THEOREM 10.15 *Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism induced by a ring map $\phi : A \to B$, and let M be an A-module. Then*

$$f^*(\widetilde{M}) = \widetilde{M \otimes_A B} \tag{10.4}$$

$$M = A^I \rightsquigarrow \widetilde{M} = \mathcal{O}_Y^I$$
$$f^*\widetilde{M} = \mathcal{O}_X^I$$

PROOF: First, note that the theorem holds in the special case when $M = A^I$ is a free module (here the index set I is allowed to be infinite) – this is simply because $f^*\mathcal{O}_Y = \mathcal{O}_X$ and f^* commutes with taking direct sums. To prove it in general, we pick a presentation of M of the form

$$A^J \xrightarrow{\gamma} A^I \to M \to 0$$

$$\widetilde{A^J} \to \widetilde{A^I} \to \widetilde{M} \to 0$$

Applying $\sim$ and then f^* (which is right-exact) we get a sequence

$$f^*\widetilde{A^J} \xrightarrow{\nu} f^*\widetilde{A^I} \to f^*\widetilde{M} \to 0$$

which is exact on the right, since f^* is right-exact. From this we get that

$$\begin{aligned} f^*\widetilde{M} &= \operatorname{Coker}\nu = \operatorname{Coker}(\widetilde{\gamma \otimes_A B}) \\ &= ((\operatorname{Coker}\gamma) \otimes_A B))^{\sim} = (M \otimes_A B)^{\sim}. \end{aligned}$$

❑

Adjoint property for quasi-coherent sheaves

Recall that we defined, for a morphism $f : X \to Y$, natural maps

$$\nu : f^* f_* \mathcal{F} \to \mathcal{F} \text{ and } \eta : \mathcal{G} \to f_* f^* \mathcal{G}$$

where $\mathcal{F} \in \mathrm{Mod}_X$ and $\mathcal{G} \in \mathrm{Mod}_Y$.

In the case of affine schemes, we can understand these maps as follows. Let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \tilde{N}$. We have $f_*\widetilde{M} = \widetilde{M}_A$ and so

$$f^* f_* \widetilde{M} = M \widetilde{_A \otimes_A} B.$$

$$f^* f_* \widetilde{M} \longrightarrow \widetilde{M}$$

$$M_A \otimes_A B \longrightarrow M$$

$$m \otimes_A b \longmapsto bm$$

The point is that since the tensor product is over the ring A, we cannot move B over to the left hand side, but we do have a natural map of B-modules $M_A \otimes_A B \to M$ given by $m \otimes b \mapsto bm$. Doing the same over each $D(f)$ we get the maps that induce ν.

$$\mathcal{G} \longrightarrow f_* f^* \mathcal{G}$$

Also, we have $f^*\widetilde{N} = N \otimes_A B$, and so

$$f_* f^* \widetilde{N} = (N \widetilde{\otimes_A B})_A.$$

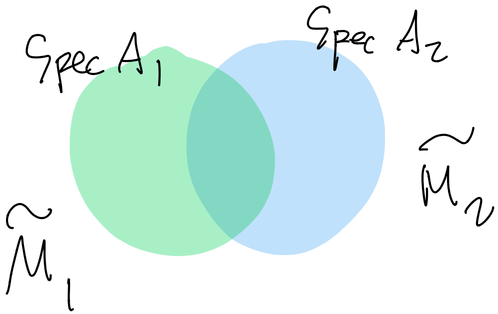
Then η is induced by the map

$$N \to (N \otimes_A B)_A$$

given by $n \mapsto n \otimes_A 1$.

Quasi-coherent sheaves on general schemes

Having established the sheaves that serve as local models for the quasi-coherent sheaves on affine schemes, we are now ready for the general definition.



Definition 10.16 *If X is a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module, one says that $\mathcal{F}$ is a* quasi-coherent *(kvasikoherent)* $\mathcal{O}_X$*-module, or* quasi-coherent sheaf *for short, if there is an open affine covering $\{U_i\}_{i \in I}$ of X, say $U_i = \operatorname{Spec} A_i$, and modules M_i over A_i such that $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$.*

Phrased in slightly different manner, $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if the restriction $\mathcal{F}|_{U_i}$ of $\mathcal{F}$ to each U_i is of type tilde of an A_i-module. In particular, the modules $\widetilde{M}$ on affine schemes $\operatorname{Spec} A$ are all quasi-coherent.

The restriction of a quasi-coherent sheaf $\mathcal{F}$ to any open set $U \subset X$ is quasi-coherent. Indeed, it will suffice to verify this for X an affine scheme, and by Lemma **??** the restriction of a sheaf of tilde-type to a distinguished open set is of tilde-type. As any open U in an affine scheme is the union of distinguished open subsets, it follows that $\mathcal{F}|_U$ is quasi-coherent.

For $\mathcal{F}$ to be quasi-coherent, we require that $\mathcal{F}$ be locally of tilde-type for just *one* open affine cover. However, it turns out that this will hold for *any* open affine cover, or equivalently, that $\mathcal{F}|_U$ is of tilde-type for any open affine subset $U \subset X$. This is a much stronger than the requirement in the definition, and it is somewhat difficult to prove. As a first corollary we arrive at the *a priori* not

$\mathcal{F}$ kvasikohent $\Leftrightarrow$ $\mathcal{F}|_{\operatorname{Spec} A} = \tilde{M}$

for alle $\operatorname{Spec} A \subset X$.

obvious conclusion that the modules of the form $\widetilde{M}$ are the only quasi-coherent $\mathcal{O}_X$-modules on an affine scheme. We shall also see that quasi-coherent modules enjoy the coherence property (**??**) on page **??** that was the point of departure for our discussion.

The story begins with a lemma that establishes the coherence property (**??**) in a very particular case; *i.e.* for sections over distinguished open sets of a quasi-coherent $\mathcal{O}_X$-module on an affine scheme $X = \operatorname{Spec} A$. For any distinguished open set $D(f) \subset X$ it holds that $\Gamma(D(f), \mathcal{O}_X) = A_f$, and consequently there is for any $\mathcal{O}_X$-module a canonical map $\Gamma(X, \mathcal{F}) \otimes_A A_f \to \Gamma(D(f), \mathcal{F})$ sending $s \otimes af^{-n}$ to $af^{-n} \cdot s|_{D(f)}$. It turns out to be an isomorphism whenever $\mathcal{F}$ is quasi-coherent:

Skal vise: $\mathcal{F}$ kvasikoherent på Spec A $\Rightarrow$ $\mathcal{F} = \widetilde{M}$

$M = \mathcal{F}(X)$! $\rightsquigarrow$ må vise at $\mathcal{F} = \widetilde{\mathcal{F}(X)}$

LEMMA 10.17 *Suppose that $X = \operatorname{Spec} A$ is an affine scheme and that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Let $D(f) \subset X$ be a distinguished open set. Then the following hold:*

- $\Gamma(D(f), \mathcal{F}) \simeq \Gamma(X, \mathcal{F})_f$.
- *Let $s \in \mathcal{F}(X)$ be a global section of $\mathcal{F}$ and assume that $s|_{D(f)} = 0$, then sufficiently large powers of f kill s, that is, for sufficiently large integers n one has $f^n s = 0$.*
- *Let $s \in \Gamma(D(f), \mathcal{F})$ be a section. Then for a sufficiently large n, the section $f^n s$ extends to a global section of $\mathcal{F}$. That is, there exists an n and a global section $t \in \Gamma(X, \mathcal{F})$ such that $t|_{D(f)} = f^n s$*

PROOF: The first statement is by the definition of localization equivalent to the two others.

The sheaf $\mathcal{F}$ is quasi-coherent by hypothesis, and the affine scheme $X = \operatorname{Spec} A$ is quasi-compact, so there is a finite open affine covering of X by distinguished sets $D(g_i)$ such that $\mathcal{F}|_{D(g_i)} \simeq \widetilde{M_i}$ for some A_{g_i}-modules M_i. The section s of $\mathcal{F}$ restricts to sections s_i of $\mathcal{F}|_{D(g_i)}$ over $D(g_i)$, that is, to elements s_i of M_i.

Further restricting $\mathcal{F}$ to the intersections $D(f) \cap D(g_i) = D(fg_i)$ yields the equality $\mathcal{F}|_{D(fg_i)} = (\widetilde{M_i})_f$, and by hypothesis, the section s restricts to zero in $\Gamma(D(fg_i), \mathcal{F}) = (M_i)_f$. This means that the localization map sends s_i to zero in $(M_i)_f$. Hence s_i is killed by some power of f, and since there is only finitely many g_i's, there is an n with $f^n s_i = 0$ for all i; that is, $(f^n s)|_{D(g_i)} = 0$ for all i. By the locality axiom for sheaves, it follows that $f^n s = 0$.

Assume now a section $s \in \Gamma(D(f), \mathcal{F})$ is given. We must show that that $f^n s$ extends to a global section of $\mathcal{F}$ for some n. Each restriction $s|_{D(fg_i)} \in \Gamma(D(fg_i), \mathcal{F}) = (M_i)_f$ is of the form $f^{-n}s_i$ with $s_i \in M_i = \Gamma(D(g_i), \mathcal{F})$, and by the usual finiteness argument, n can be chosen uniformly for all i. This means that $s_i = f^n s$ and $s_j = f^n s$ mach on the intersection $D(f) \cap D(g_i) \cap D(g_j)$, and by the first part of the lemma applied to Spec $A_{g_i g_j}$, one has $f^N(s_i - s_j) = 0$ on $D(g_i) \cap D(g_j)$ for a sufficiently large integers N. Hence the different $f^N s_i$'s patch together to give the desired global section t of $\mathcal{F}$. ❑

Theorem 10.18 *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for all open affine subsets $U \subset X = \operatorname{Spec} A$, the restriction $\mathcal{F}|_U$ is isomorphic to an $\mathcal{O}_X$-module of the form $\widetilde{M}$ for an A-module M.*

$U = \operatorname{Spec} A$

$$\beta : \widetilde{M} \longrightarrow \mathcal{F} \qquad M = \mathcal{F}(X)$$

$$m/f^n \longrightarrow m \cdot f^{-n}|_{D(f)}$$

Proof: As quasi-coherence is conserved when restricting $\mathcal{O}_X$-modules to open sets, we may surely assume that X itself is affine; say $X = \operatorname{Spec} A$. Let $M = \mathcal{F}(X)$. We saw in Lemma **??** on page **??** that there is a natural map $\beta : \widetilde{M} \to \mathcal{F}$ that on distinguished open sets sends mf^{-n} to $f^{-n}m|_{D(f)}$. But by the fundamental lemma **??** above, this is an map is both injective and surjective over the distinguished open sets. Hence the two sheaves are isomorphic. ❑

Applying this to an affine scheme, yields the important fact that any quasi-coherent sheaf (in the sense of Definition **??**) $\mathcal{F}$ on an affine scheme $X = \operatorname{Spec} A$ is of the form $\widetilde{M}$ for an A-module M.

$$\mathrm{Mod}_A \longrightarrow \mathrm{QCoh}_X$$

$$M \longmapsto \widetilde{M}$$

$$\mathcal{F}(X) \longleftarrow \mathcal{F}$$

Proposition 10.19 *Assume that* $X = \operatorname{Spec} A$. *The tilde-functor* $M \mapsto \widetilde{M}$ *is an equivalence of categories* Mod_A *and* QCoh_X *with the global section functor as an inverse.*

Theorem 10.20 *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Then $\mathcal{F}$ is quasi-coherent if and only if for any pair $V \subset U$ open affine subsets, the natural map*

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V) \tag{10.5}$$

is an isomorphism.

$V = D(g)$

$U = D(f)$

$\mathcal{F} = \widetilde{M}$

$\rightsquigarrow \quad M_f \otimes_{A_f} A_g \xrightarrow{\sim} M_g$

$= (M_f)_g$

PROOF: We may clearly assume that X is affine, say $X = \operatorname{Spec} A$.

Assume first that the maps (**??**) are isomorphisms. We may take $V = D(f)$ and $U = X$ and $M = \mathcal{F}(X)$. Then from (**??**) it follows that $\Gamma(D(f), \mathcal{F}) = M_f$ which shows that the canonical map $\beta : \widetilde{M} \to \mathcal{F}$ is an isomorphism over all distinguished open subsets, and therefore an isomorphism. Hence $\mathcal{F}$ is quasi-coherent.

$$i : \operatorname{Spec} B \hookrightarrow \operatorname{Spec} A = X$$

$$i^* \widetilde{M} = \widetilde{M \otimes_A B}$$

To argue for the reverse implication, we may again assume $X = \operatorname{Spec} A$, $U = X$ and $V = \operatorname{Spec} B$. So suppose that $\mathcal{F}$ is quasi-coherent; that is, $\mathcal{F} = \widetilde{M}$ for some A-module M after theorem **??**. Let $i : V \to X$ denote the inclusion map. We have $i^* \widetilde{M} = \widetilde{M}|_U \simeq \widetilde{M \otimes_A B}$. Taking global sections, we get exactly the map in **??**) which is then an isomorphism. ❑

$$\Gamma(V, i^* \widetilde{M}) \xleftarrow{\sim} \Gamma(\widetilde{M \otimes_A B})$$

$$\Gamma(V, i^* \widetilde{M}) = \mathcal{F}(V)$$

$$\Gamma(\widetilde{M \otimes_A B}) = M \otimes_A B = \mathcal{F}(U) \otimes_{\mathcal{O}_{X}(U)} \mathcal{O}(V)$$

Example 10.21 The example of an discrete valuation ring is always useful to consider, and we continue exploring Example **??** above. Consider the $\mathcal{O}_X$-module $\mathcal{F}$ given by the data M, N, ρ. We claim that $\mathcal{F}$ is quasi-coherent if and only if ρ is an isomorphism.

$$\rho : M \otimes_R K \longrightarrow N$$

$U = X$ $\quad$ $\mathcal{F}(U) = M$

$V = \{\eta\}$ $\quad$ $\mathcal{F}(V) = N$

$M \otimes_R K \longrightarrow N$

$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}(V) \longrightarrow \mathcal{F}(V)$

If $\mathcal{F}$ is quasi-coherent, then every point has a neighbourhood on which $\mathcal{F}$ is the $\tilde{}$ of some module. The only neighbourhood of the unique closed point is X itself, and so $\mathcal{F} = \widetilde{M}$. Therefore, $N = \mathcal{F}(U) = M_{(0)} = M \otimes_R K$ and ρ is an isomorphism. Conversely, of $\rho : M \otimes_R K \to K$ is an isomorphism, then $\mathcal{F}$ is given by $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = M \otimes_R N$, and so $\mathcal{F} \simeq \widetilde{M}$ is quasi-coherent.

☆

10.4 *Coherent sheaves*

Let A be a ring and let M be an A-module. The module M is *of finite presentation* if for some integers n and m there is an exact sequence

$$A^n \longrightarrow A^m \longrightarrow M \longrightarrow 0 \,.$$

One says that M is *coherent* if the following two requirements are fulfilled

- M is finitely generated.
- The kernel of every surjection $A^n \to M$ is finitely presented.

- The kernel of every surjection $A^n \to M$ is finitely presented.

When A is noetherian, then the three conditions of *coherence*, *finitely generation* and *being of finite presentation* on an A-module M coincide. The key point is that for a finitely generated module M over a noetherian ring A, every submodule $N \subset M$ is also finitely generated. Indeed, the set $\mathcal{V}$ of finitely generated submodules $N' \subset N$ has a maximal element N°, which has to equal N: If not, there is an $n \in N - N^\circ$, and a finitely generated submodule $N'' = N' + An$ which is strictly bigger than N°.

So to show that M is finitely presented, we can take a presentation $A^m \to M \to 0$ and let L be the kernel, regarded as a submodule of A^n. Then L is finitely generated, so there is a surjection $A^n \to L$, and we get a presentation $A^n \to A^m \to M \to 0$. Applying this argument to any finitely generated submodule $N \subset M$ shows that M is coherent.

Definition 10.22 *On a scheme X a quasi-coherent $\mathcal{O}_X$-module is* coherent *if there is a covering of X by open affine sets $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} = \widetilde{M}_i$ with the M_i's being coherent A_i-modules. $\mathcal{F}$ is finitely presented if each M_i are finitely presented as A_i-modules.*

So if X is noetherian (or locally noetherian), the condition that M is coherent is equivalent to the weaker condition that M is finitely generated.

10.5 Functoriality

Proposition 10.24 *Suppose that $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent sheaves on the scheme X.*

- *The kernel, cokernel and the image of α are all quasi-coherent.*
- *The category QCoh_X is closed under extensions; that is, if*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{10.6}$$

is a short exact sequence of $\mathcal{O}_X$-modules with the two extreme sheaves M' and M'' being quasi-coherent, the middle sheaf M is quasi-coherent as well.

PROOF: If $\alpha\colon \mathcal{F} \to \mathcal{G}$ is a map of quasi-coherent $\mathcal{O}_X$-modules, on any open affine subsets $U = \operatorname{Spec} A$ of X it may be described as $\alpha|_U = \widetilde{a}$ where $a\colon M \to N$ is a A-module homomorphism and M and N are A-modules with $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Since the tilde-functor is exact, one has $\operatorname{Ker} \alpha|_U = (\operatorname{Ker} a)\widetilde{\ }$. Moreover, by the same reasoning, it holds true that $\operatorname{Coker} \alpha|_U = (\operatorname{Coker} a)\widetilde{\ }$ and $\operatorname{Im} \alpha|_U = (\operatorname{Im} a)\widetilde{\ }$.

Suppose now that an extension like (10.6) is given. The leftmost sheaf M' being quasi-coherent, Lemma B.4 shows that the induced sequence of global sections is exact; that is, the upper horizontal sequence in the diagram below. The three vertical maps in the diagram are the natural maps from Lemma 10.9 on page 170. Since M' and M'' both are quasi-coherent sheaves, the two flanking vertical maps are isomorphisms, and the snake lemma implies that the middle vertical map is an isomorphism as well. Hence M is quasi-coherent.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X, M')^{\sim} & \longrightarrow & \Gamma(X, M)^{\sim} & \longrightarrow & \Gamma(X, M'')^{\sim} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

❑

Thus also for a general scheme X, the category QCoh_X is a category with very nice properties: it is an abelian category with tensor products and internal Hom's.

Quasi-coherence of pullbacks

Recall that for a morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes, the pullback of the quasi-coherent sheaf $\widetilde{M}$ was again quasi-coherent: This followed from the formula in Theorem 10.15

$$f^*(\widetilde{M}) = \widetilde{M \otimes_A B}$$

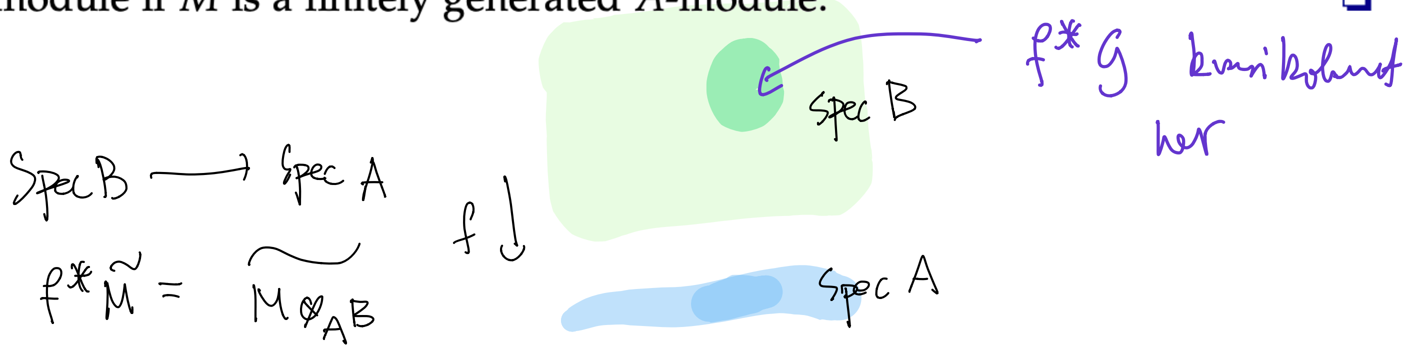
In this section, we prove that the same conclusion holds more generally.

Proposition 10.25 *Let $f : X \to Y$ be a morphism of schemes.*

i) *If $\mathcal{G}$ is a quasi-coherent sheaf on Y, then $f^*\mathcal{G}$ is quasi-coherent on X.*

ii) *If moreover X and Y are noetherian, then $f^*\mathcal{G}$ is coherent if $\mathcal{G}$ is.*

Proof: These statements follows from the affine case and the formula above, since quasi-coherence is a local property, and since $M \otimes_A B$ is a finitely generated B-module if M is a finitely generated A-module. ❑



Quasi-coherence of pushforwards

Likewise, we showed that for a map $\phi : \operatorname{Spec} A \to \operatorname{Spec} N$, the pushforward $\phi_* \mathcal{F}$ is quasi coherent, if $\mathcal{F}$ is quasi-coherent (since $\phi_* \widetilde{M} = \widetilde{M_B}$). The following result applies to more general morphisms:

THEOREM 10.26 *Let $\phi \colon X \to Y$ be a morphism of schemes and that $\mathcal{F}$ is a quasi-coherent sheaf on X. If X is noetherian, then the direct image $\phi_* \mathcal{F}$ is quasi-coherent on Y.*

Proof: We may assume that $Y = \operatorname{Spec} A$. Then since X is quasi-compact, we may cover it by open affines U_i. The intersection $U_i \cap U_j$ is again quasi-compact, so we can cover it with open affines U_{ijk}.

For any open $V \subset Y$, one has the exact sequence

$$0 \longrightarrow \Gamma(\phi^{-1}V, \mathcal{F}) \longrightarrow \prod_i \Gamma(U_i \cap \phi^{-1}V, \mathcal{F}) \longrightarrow \prod_{i,j,k} \Gamma(U_{ijk} \cap \phi^{-1}V, \mathcal{F}). \tag{10.7}$$

(10.7)

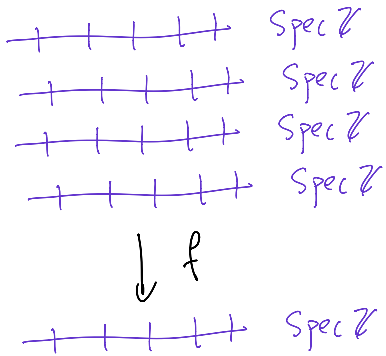
The sequence is compatible with the restriction maps induced from an inclusion $V' \subset V$, hence gives rise to the following exact sequence of sheaves on X:

$$0 \longrightarrow \phi_* \mathcal{F} \longrightarrow \prod_i \phi_{i*} \mathcal{F}|_{U_i} \longrightarrow \prod_{i,j,k} \phi_{ijk_*} \mathcal{F}|_{U_{ijk}} \tag{10.8}$$

where $\phi_i = \phi|_{U_i}$ and $\phi_{ijk} = \phi|_{U_{ijk}}$. Now, each of the sheaves $\phi_{i*}\mathcal{F}|_{U_i}$ and $\phi_{ij_*}\mathcal{F}|_{U_{ij}}$ are quasi-coherent by the affine case of the theorem (proposition 10.14 on page 173). They are finite in number as the covering U_i is finite. Hence $\prod_i \phi_{i*}\mathcal{F}|_{U_i}$ and $\prod_{i,j} \phi_{ij_*}\mathcal{F}|_{U_{ij}}$ are finite products of quasi-coherent $\mathcal{O}_X$-modules and therefore they are quasi-coherent. Now the $\phi_*\mathcal{F}$ is the kernel of a homomorphism between two quasi-coherent sheaves, and so the theorem the follows from Proposition 10.24 on page 179. ❑

The following example shows that some of the hypotheses are neccessary for this statement to hold:

Example 10.27 Let $X = \coprod_{i \in I} \operatorname{Spec} \mathbb{Z}$ be the disjoint union of countably infinitely many copies of $\operatorname{Spec} \mathbb{Z}$ and let $f\colon X \to \operatorname{Spec} \mathbb{Z}$ be the morphism that equals the



identity on each of the copies of $\operatorname{Spec}\mathbb{Z}$ that constitute X. Then $f_*\mathcal{O}_X$ is not quasi-coherent. Indeed, the global sections of $f_*\mathcal{O}_X$ satisfy

$$\Gamma(\operatorname{Spec}\mathbb{Z}, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i\in I}\mathbb{Z}.$$

$\mathcal{F}$ qc $\iff$ $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}(V) \xrightarrow{\sim} \mathcal{F}(V)$

$U \supseteq V$

$U = \operatorname{Spec} \mathbb{Z}$ $\quad V = D(p)$

On the other hand if p is any prime, one has

$$\Gamma(D(p), f_*\mathcal{O}_X) = \Gamma(f^{-1}D(p), \mathcal{O}_X) = \prod_{i \in I} \mathbb{Z}[p^{-1}].$$

It is not true that $\Gamma(D(p), f_*\mathcal{O}_X) = \Gamma(\operatorname{Spec} \mathbb{Z}, f_*\mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ hence $f_*\mathcal{O}_X$ is not quasi-coherent. Indeed, elements in $\prod_{i \in I} \mathbb{Z}[p^{-1}]$ are sequences of the form $(z_i p^{-n_i})_{i \in I}$ where $z_i \in \mathbb{Z}$ and $n_i \in \mathbb{N}$. Such an element lies in $(\prod_{i \in I} \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ only if the n_i's form a bounded sequence, which is not the case for general elements of shape $(z_i p^{-n_i})_{i \in I}$ when I is infinite. ☆

$(1, p^{-1}, p^{-2}, \ldots)$

$\sum n_i \otimes \frac{m_i}{p^n}$

endlich $\mathbb{Z}$

Coherence of the direct image

For morphisms of schemes $f : X \to Y$, it cannot be expected that the push-forward of a coherent sheaf is again coherent, even for 'nice' morphisms f. A simple example is the following:

Example 10.28 Let $X = \operatorname{Spec} k[t]$ and consider the structure morphism $f : X \to \operatorname{Spec} k$ (induced by $k \subset k[t]$). The sheaf $\mathcal{O}_X$ is of course coherent, but $f_*\mathcal{O}_X$ is not. Indeed, this is $\widetilde{k[t]}$, and $k[t]$ is clearly not finitely generated as a k-module. ☆

$$X = \operatorname{Spec} k[t] \xrightarrow{f} \operatorname{Spec} k = Y$$

$$f_*\mathcal{O}_X = f_*\widetilde{k[t]} = \widetilde{k[t]} \leftarrow \text{sett på som } k\text{-modul.}$$

som $k[t]$-modul

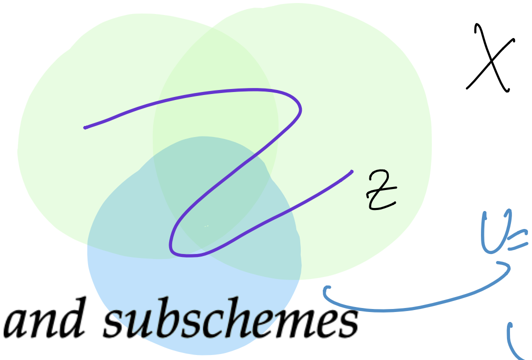
However, for *finite* morphisms, we have a positive result:

Lemma 10.29 *Let $f : X \to Y$ be a finite morphism of schemes. If $\mathcal{F}$ be a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent on Y. If X and Y are noetherian, $f_*\mathcal{F}$ is even coherent if $\mathcal{F}$ is.*

M end gen. B-modul

$A \to B$ endelig $\implies$ M_A end. gen A-modul

PROOF: Since f is finite, we can cover Y by open affines $\operatorname{Spec} A$ such that each $f^{-1} \operatorname{Spec} A = \operatorname{Spec} B$ is also affine, where B is a finite A-module. We then have $f_*\mathcal{F}(\operatorname{Spec} A) = \mathcal{F}(\operatorname{Spec} B)$. Now, since $\mathcal{F}$ is quasicoherent, we have $\mathcal{F}|_{\operatorname{Spec} B} = \widetilde{M}$ for some B-module, which we can view as an A-module via f. Hence $f_*\mathcal{F}$ is quasi-coherent. If X and Y are noetherian, and $\mathcal{F}$ is coherent, the module M is finitely generated as an B-module, and hence as an A-module, since B is a finite A-module. ❑



$U = \operatorname{Spec} A$

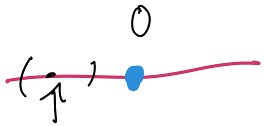
$U \cap Z$ " = " $\operatorname{Spec} A/I$ "

10.7 Closed immersions and subschemes

Recall that for a scheme X, a *closed subscheme* was given by a closed subset $Z \subset X$ equipped with a sheaf of rings $\mathcal{O}_Z$ making $(Z, \mathcal{O}_Z)$ into a scheme, and so that $i_* \mathcal{O}_Z \simeq \mathcal{O}_X / \mathcal{I}$ for some sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. In Chapter 2, we considered the prototype example namely when $X = \operatorname{Spec} A$ and $Z = V(I)$ for some ideal $I \subset A$; then the closed subscheme Z is isomorphic to $\operatorname{Spec}(A/I)$. However, it was not clear which ideal sheaves gave rise to closed subschemes, even in the case for affine schemes. In this section, we will show that the right condition is that the ideal sheaf should be quasi-coherent.

ex $X = \mathbb{A}^1 = \operatorname{Spec} k[t]$ $\qquad 0 \in X$

$$\mathcal{I}(U) = \begin{cases} 0 & U \not\ni 0 \\ \mathcal{O}_X(U) & U \ni 0 \end{cases}$$



$$\rightsquigarrow \quad \operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) = \mathbb{A}^1 - 0$$

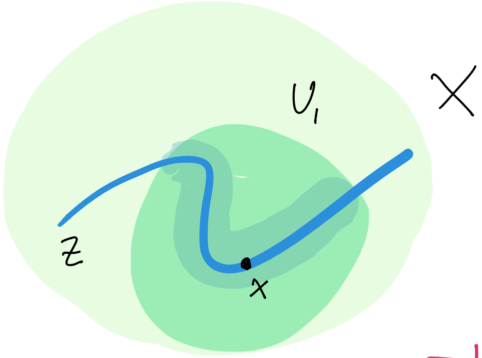
$$\{x \in X \mid (\mathcal{O}_X/\mathcal{I})_x \neq 0\}$$

!

Lemma 10.32 *Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. Then the ringed space $Z = (\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ is a scheme with a canonical morphism $i : Z \to X$.*

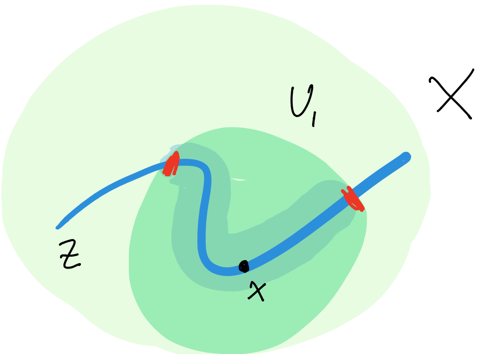
PROOF: To prove this, we may assume that $X = \operatorname{Spec} A$ is affine. In this case $\mathcal{I}$ is the $\sim$ of some ideal $I \subset A$, and the support of this is exactly the primes $\mathfrak{p}$ such that $(A/I)_{\mathfrak{p}} \neq 0$, or equivalently $\mathfrak{p} \in V(I)$. Hence Z is the closed subset $V(I)$, which is homeomorphic to $\operatorname{Spec}(A/I)$. The sheaf of rings on $\operatorname{Spec}(A/I)$ is the same as $\mathcal{O}_X/\mathcal{I}$ on Z and hence Z is the scheme $\operatorname{Spec} A/I$. The map i is just induced by the inclusion $Z \subset X$ and the natural map $\mathcal{O}_X \to i_*(\mathcal{O}_X/\mathcal{I})$ is just $\sim$ of the quotient map $A \to A/I$. ❑

Proposition 10.33 *Let $Z \subset X$ is a closed subscheme of X, given by an ideal sheaf $\mathcal{I}$, then $\mathcal{I}$ is quasi-coherent.*



$\mathcal{I}|_{X-Z} = \mathcal{O}_{X-Z}$

PROOF: On the open set $X - Z$, we have $\mathcal{I} \cong \mathcal{O}_X$, so it is quasi-coherent there. Let $x \in Z$. We first find an affine open set $U = \operatorname{Spec} A$ of x such that $U \cap Z$ is an open affine in Z. To find U, pick any affine open set $U_1 \subset X$ and let $V_1 \subset U_1 \cap Z$ be an affine open set containing x. Then pick a section $s \in \mathcal{O}_X(U_1)$ so that $s = 0$ on $U_1 \cap Z - V_1$, while $s(x) \neq 0$. Then let $U = D(s) \subset U_1$. Note



that $U \cap Z = D(s|_{V_1}) \subset V_1$, so $U \cap Z$ is an affine subset in Z as well. Write $U = \operatorname{Spec} A$ and $U \cap Z = \operatorname{Spec} B$, so that the inclusion $U \cap Z \to U$ corresponds to $\phi : A \to B$. Let $I = \operatorname{Ker} \phi$. We claim that

"$B = A/I$"

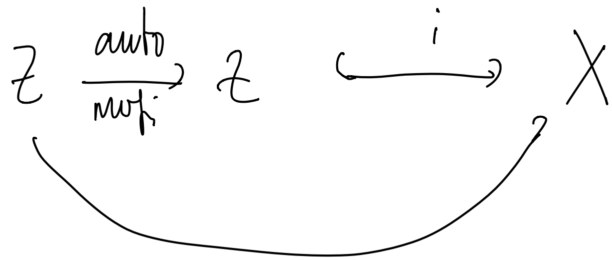
$$\mathcal{I}|_U \simeq \tilde{I}$$

so that $\mathcal{I}$ is indeed quasi-coherent. Indeed, for any distinguished open set $D(f)$

in U we have

$$\begin{aligned}\widetilde{I}(D(f)) &= I_f \\ &= \mathrm{Ker}(A_f \to B_f) \\ &= \mathrm{Ker}(\mathcal{O}_U(D(f)) \to \mathcal{O}_Z(Y \cap D(f))) \\ &= \mathcal{I}(D(f))\end{aligned}$$

This completes the proof. ❑



Notice that the closed subset Z can be recovered from the ideal sheaf $\mathcal{I}$ by $Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$. In particular, this gives the most economic way of defining what a closed subscheme of X is: It is a subscheme of the form $(\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ for some quasi-coherent sheaf of ideals $\mathcal{I}$.

Now we can finally prove Proposition 3.19 from Chapter 2.

Corollary 10.34 *Let $Z \subset X$ is a closed subscheme given by an ideal sheaf $\mathcal{I}$, then for all open affines $U \subset X$, $U \cap Z$ is affine in Z. Moreover, if $U = \operatorname{Spec} A$, then $Z \cap U \simeq \operatorname{Spec}(A/I)$ for some ideal $I \subset A$.*

PROOF: Since $\mathcal{I}$ is quasi-coherent, we have $\mathcal{I} = \widetilde{I}$ for some ideal $I \subset A$. Then we get

$$\begin{aligned} \mathcal{O}_Z|_U &= \operatorname{Coker}(\mathcal{I}|_U \to \mathcal{O}_X|_U) \\ &= \operatorname{Coker}(\widetilde{I} \to \widetilde{A}) \\ &= \widetilde{A/I} \end{aligned}$$

It follows that $(Y, \mathcal{O}_Y) = (V(I), \widetilde{A/I}) = \operatorname{Spec}(A/I)$. ❑

∴ alle underskjemaer av Spec A
er gitt ved idealer $I \subset A$.

Corollary 10.35 *Let* $X = \operatorname{Spec} A$ *be an affine scheme. Then the map* $I \mapsto V(I)$ *induces a 1-1 correspondence between the set of ideals of* A *and closed subschemes of* X*. In particular, any closed subscheme of an affine scheme is also affine.*

Morphisms to a closed subscheme

If $f : Y \to X$ is a morphism of schemes, it is natural to ask when it factors through a closed immersion of X. Here we need only work up to isomorphism, so we can assume that the closed immersion is given by $(\operatorname{Supp} \mathcal{O}_X/\mathcal{I}, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)$.

Proposition 10.36 *Let Z be a closed subscheme of X given by sheaf of ideals $\mathcal{I}$. Suppose $f : Y \to X$ is a morphism of schemes. Then f factors through a map $g : Y \to Z$ if and only if*

i) $f(Y) \subset Z$.

ii) $\mathcal{I} \subset \mathrm{Ker}(\mathcal{O}_X \to f_*(\mathcal{O}_Y))$.

PROOF: The condition (i) is clearly necessary. If there is a sequence $Y \to Z \to X$, then there is a sequence of sheaves $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to f_*(\mathcal{O}_Y)$, which means that the map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$ factors through $\mathcal{O}_X/\mathcal{I}$, and so also (ii) holds.

Conversely, we define the map g on topological spaces by the inclusion (i). To define it on sheaves, we use the map $\mathcal{O}_X \to f_*(\mathcal{O}_Y)$. This annihilates $\mathcal{I}$, so we thus get a map $\mathcal{O}_X/\mathcal{I} \to f_*(\mathcal{O}_Y) = g_*(\mathcal{O}_Y)$. This gives us the map $g : Y \to Z$ factoring f. ❑

For a morphism of schemes $f : Y \to X$, we can define the *scheme-theoretic image* of f as a subscheme $Z \subset X$ satisfying the universal property that if f factors through a subscheme $Z' \subset Z$, then $Z \subset Z'$. To define Z it is is tempting to use the ideal sheaf $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_X \to f_*(\mathcal{O}_Y))$ – but this may fail to be quasi-coherent for a general morphism f. However, one can show that there is a largest quasi-coherent sheaf of ideals $\mathcal{J}$ contained in $\mathcal{I}$, and we then define Z to be associated to $\mathcal{J}$.