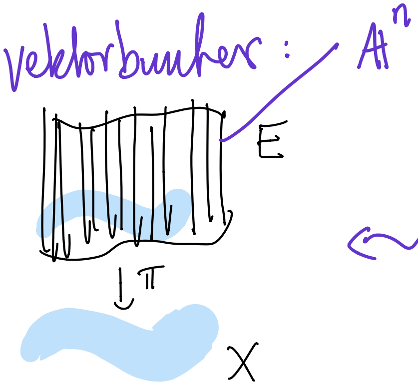
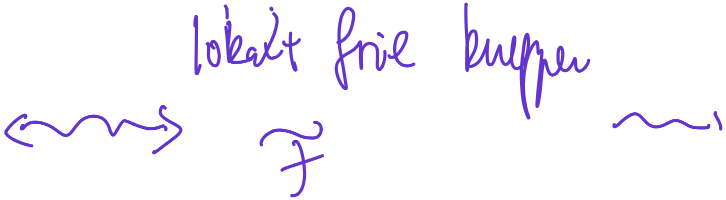
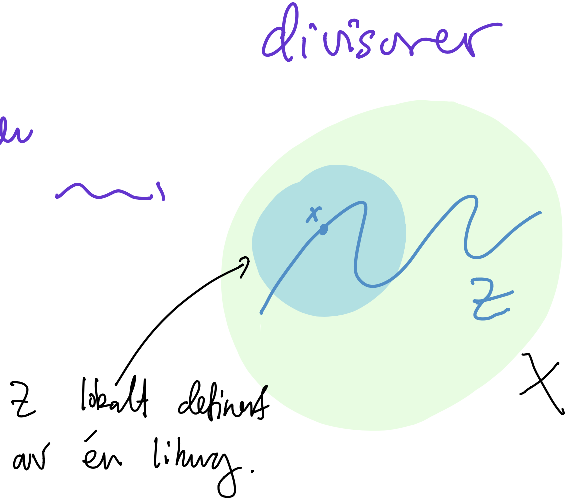






Chapter 11

Locally free sheaves

$$\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r_i} \qquad r_i = \operatorname{rk} \mathcal{E}$$

An $\mathcal{O}_X$-module $\mathscr{E}$ is called *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is *locally free* if there exists a *trivializing cover*, that is, an open cover $\{U_i\}_{i \in I}$ such that $\mathscr{E}|_{U_i}$ is free for each i. The *rank* of $\mathscr{E}$ at a point $x \in U_i$ is the number of copies of $\mathcal{O}_{U_i}$ needed (this may be finite or infinite). If X is connected, then the rank of $\mathscr{E}$ is the same everywhere, but in general we allow variation. A locally free sheaf of rank 1 is called an *invertible sheaf*.

← kanskje de viktigste $\mathcal{O}_X$-modulene.

Example 11.1 The sheaf $\mathcal{O}_X^r = \bigoplus_{i=1}^r \mathcal{O}_X$ is a locally free sheaf of rank r. As this is globally a free sheaf, it is sometimes called 'trivial'. ☆

If $\mathscr{E}$ is a locally free sheaf, the stalk $\mathscr{E}_x$ is a free $\mathcal{O}_{X,x}$-module for every $x \in X$. In fact, under some coherence conditions, the converse holds:

LEMMA 11.2 *Suppose that X is locally noetherian. A coherent sheaf $\mathscr{E}$ having the property that $\mathscr{E}_x \simeq \mathcal{O}_{X,x}^r$ for every $x \in X$ for some fixed r, is locally free.*

Locally free sheaves and projective modules

On an affine scheme $X = \operatorname{Spec} A$, any quasi-coherent $\mathcal{O}_X$-module $\mathscr{E}$ is isomorphic to $\widetilde{M}$ for some A-module M. Thus a natural question is which A-modules give rise to locally free sheaves. The main result of this section is that $\mathscr{E}$ is locally free of finite rank if and only if M is finitely generated and projective.

Recall that an A-module M is called *projective* if there is another module N so that $M \oplus N \simeq A^r$ is free. M being projective can further be characterized as saying that the functor $N \mapsto \mathrm{Hom}_A(M, N)$ is exact. It is clear that free modules satisfy this property, but there are many examples of projective modules which are not free.

Hom$(M, -)$ eksakt

LEMMA 11.4 *Let $(A, \mathfrak{m})$ be a local ring and M a finitely generated projective module. Then M is free.*

Følger av Nakayama's lemma.

Proposition 11.5 *Let $X = \operatorname{Spec} A$ where A is Noetherian, and let $\mathcal{F} = \widetilde{M}$ be a coherent sheaf. The following are equivalent:*

i) $\mathcal{F}$ *is locally free;*

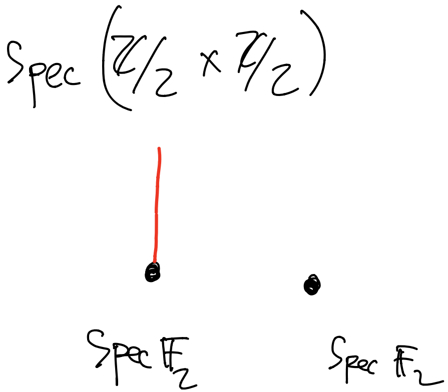
ii) $\mathcal{F}_x$ *is a free* $\mathcal{O}_{X,x}$*-module for all* $x \in X$*;*

iii) M *is* locally free, *i.e.,* $M_{\mathfrak{p}}$ *is free for all* $\mathfrak{p} \in \operatorname{Spec} A$*;*

iv) M *is* projective, *i.e. there is a module* N *such that* $M \oplus N \simeq A^I$ *is free.*

Proof: This is really a result in commutative algebra, so we only say a few words. We have already seen the equivalence *i)* $\iff$ *ii)*. The equivalence *ii)* $\iff$ *iii)* follows by definition of $\widetilde{M}$, and finally, *iii)* $\Rightarrow$ *iv)* follows because being 'projective' is a local property, *i.e.* M is projective if and only if $M_{\mathfrak{p}}$ is for every $\mathfrak{p} \in \operatorname{Spec} A$. The implication *iv)* $\Rightarrow$ *iii)* follows from the lemma above. ❑

Example 11.6 Let $X = \operatorname{Spec} A$ where $A = \mathbb{Z}/2 \times \mathbb{Z}/2$ and consider the module $M = \mathbb{Z}/2 \times (0)$ which has the structure of an A-module. Then M is projective,



$\rightsquigarrow$ M projektiv:

$M = \mathbb{Z}/2 \times (0)$

$N = 0 \times \mathbb{Z}/2$

$\rightsquigarrow M \oplus N = A$

since if $N = (0) \times \mathbb{Z}/2$, we have $M \otimes N \simeq A$ (as A-modules!). However, M is clearly not free, since any free A module must have at least four elements! The sheaf $\mathscr{E} = \tilde{M}$ is thus locally free, but not free on X. Note that X consists of two copies of $\operatorname{Spec} \mathbb{Z}/2$. $\mathscr{E}$ restricts to the structure sheaf on one of these and to the zero sheaf on the other. $\star$

$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$

$[x_0, x_1] \longrightarrow [x_0^2, x_1^2]$

endelig?

$f_* \mathcal{O}_{\mathbb{P}^1}$ er lokalt fri, men ikke fri.

EXAMPLE 11.8 Consider the morphism $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ given by $k[u] \mapsto k[t]$ $u \mapsto t^2$ on $U = \operatorname{Spec} k[t]$, and $k[u^{-1}] \to k[t^{-1}]$ $u^{-1} \mapsto t^{-2}$ on $V = \operatorname{Spec} k[t^{-1}]$. Then $f_*\mathcal{O}_{\mathbb{P}^1}$ is a locally free sheaf. Indeed, on U, $f_*\mathcal{O}_{\mathbb{P}^1}|_U$ is the $\sim$ of $k[t]$ as a $k[u]$-module, which equals $k[u] \oplus k[u]t$. We get a similar expression on $V = \operatorname{Spec} k[t^{-1}]$. It follows that $f_*\mathcal{O}_{\mathbb{P}^1}$ is locally free of rank 2. We will se in Chapter 12 just what this sheaf is, and that it is indeed not isomorphic to $\mathcal{O}^2_{\mathbb{P}^1}$. ☆

$U = \operatorname{Spec} k[t]$

$f|_U : \operatorname{Spec} k[t] \longrightarrow \operatorname{Spec} k[u]$

$t^2 \longleftarrow u$

$k[u] \longrightarrow k[t]$

$u \longrightarrow t^2$

$f_* \mathcal{O}_{\mathbb{P}^1}|_U = f_* \widetilde{k[t]}$

$= \widetilde{k[t]} \leftarrow k[u]$ modul

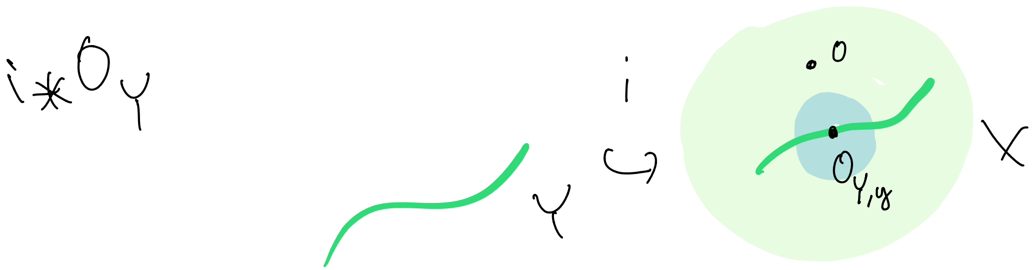
$= \widetilde{k[u] \oplus k[u]t}$

$$= \widetilde{k[u]} \oplus \widetilde{k[u]t}$$

$$= \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{F}$$

$$\mathcal{F} \not\cong \mathcal{O}_{\mathbb{P}^1}$$

A word of warning: the pushforward of a locally free sheaf is not locally free in general. For instance, if $i : Y \to X$ is a closed immersion, then $\mathcal{F} = i_* \mathcal{O}_Y$ has $\mathcal{F}_y = \mathcal{O}_{Y,y}$ for $y \in Y$, but zero stalks outside of Y. However, in general, if $\mathcal{F}$ is locally free of positive rank, then $\text{Supp}(\mathcal{F}) = X$.

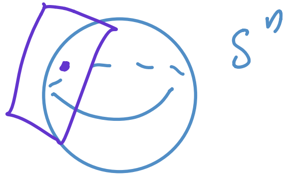


For pullbacks however, we have the following:

LEMMA 11.9 *Let $f\colon X \to Y$ be a morphism of schemes. If $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is a locally free $\mathcal{O}_X$-module.*

PROOF: Let U_i be trivialization of $\mathcal{G}$ on Y, such that $\mathcal{F}|_{U_i} \simeq \bigoplus_I \mathcal{O}_{U_i}$. Then, since $f^*\mathcal{O}_Y = \mathcal{O}_X$, we see that $f^{-1}(U_i)$ is a trivialization of $f^*\mathcal{G}$. ❑

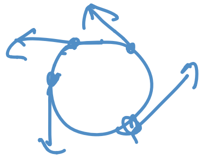
$0 \to M \to A^{n+1} \to A \to 0$



EXAMPLE 11.10 (The tangent bundle of the n-sphere) Let $X = \operatorname{Spec} A$ where $A = \mathbb{R}[x_0, \ldots, x_n]/(x_0^2 + \cdots + x_n^2 - 1)$, and consider the A-module homomorphism $f : A^{n+1} \to A$ given by $f(e_i) = x_i$. Then $M = \operatorname{Ker} f$ gives rise to a quasi-coherent sheaf $\mathscr{T} = \widetilde{M}$. Any element in the kernel corresponds to a vector of elements $v = (a_0, \ldots, a_n) \in A^{n+1}$ so that

$$a_0 x_0 + \cdots + a_n x_n = 0$$

On $U = D(x_0)$ we may divide by x_0, and solve for a_0, so v is uniquely determined by the elements $(a_1, \ldots, a_n)$. Conversely, given any such an n-tuple of elements in A, we may define an element $v \in M_{x_0}$ using the above relation. In particular, $M_{x_0} \simeq A^n$. A similar argument works for the other x_i, showing that $\mathcal{F}$ is locally free. It is a hard theorem that $\mathscr{T}$ is not free, if $n \notin \{0, 1, 3, 7\}$. ☆



$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{Q}$

The dual of a locally free sheaf

Given a locally free sheaf $\mathscr{E}$ of finite rank, we define the *dual* as

$$\mathscr{E}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X).$$

We note that $\mathscr{E}^\vee$ is again a locally free sheaf on X, of the same rank as $\mathscr{E}$. Indeed, if U is an open set in a trivializing cover, then we have

$$\begin{aligned}\mathscr{E}^\vee|_U &= \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)|_U = \mathscr{H}om_{\mathcal{O}_U}(\mathscr{E}|_U, \mathcal{O}_U) \\ &= \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_U^r, \mathcal{O}_U) \simeq \mathcal{O}_U^r.\end{aligned}$$

where the last isomorphism sends $\phi : \mathcal{O}_U^r(W) \to \mathcal{O}_U(W)$ to the tuple $(\phi(e_1), \ldots, \phi(e_r))$, where e_i is the i-th basis vector in $\mathcal{O}_U^r$. The dual operation satisfies $(\mathscr{E}^\vee)^\vee = \mathscr{E}$.

Proposition 11.11 *Let $\mathscr{E}$ be a locally free sheaf of finite rank. Then for any $\mathcal{O}_X$-module $\mathcal{F}$, we have a natural isomorphism*

$$\mathscr{E}^\vee \otimes \mathcal{F} \to \mathscr{H}om(\mathscr{E}, \mathcal{F}) \tag{11.1}$$

Proof: If A is a ring, and M, N are A-modules, we have a morphism $\mathrm{Hom}_A(M, A) \otimes N \to \mathrm{Hom}_A(M, N)$ given by $\phi \otimes n \mapsto \phi(-)n$. When $M = A^r$, any element of $\mathrm{Hom}_A(M, A) \otimes N$ can be written as $\phi_1 \otimes n_1 + \cdots + \phi_r \otimes n_k$ where $\phi_i : M \to A$ is the i-th coordinate map, and it is easy to see that the previous map is an isomorphism.

We can similarly define a sheaf map

$$\phi : \mathscr{H}om(\mathscr{E}, \mathcal{O}_X) \otimes \mathcal{F} \to \mathscr{H}om(\mathscr{E}, \mathcal{F})$$

by setting, for each open set $U \subset X$, ϕ_U to be the map

$$\phi_U : \mathrm{Hom}_{\mathcal{O}_U}(\mathscr{E}|_U, \mathcal{F}|_U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to \mathrm{Hom}_{\mathcal{O}_U}(\mathscr{E}|_U, \mathcal{F}_U)$$

sending $\phi \otimes t$ to the sheaf map $\mathscr{E}|_U \to \mathcal{F}|_U$ given by sending $s \in \mathscr{E}(V)$ to $\phi_V(s)t|_V$ over an open set $V \subset U$. This defines the sheaf map on presheaves, so sheafifying we get the map (11.1). When $\mathscr{E}$ is locally free, ϕ is an isomorphism, because it is an isomorphism on stalks, by the above. ❑

Exercise 11.3 *(The Projection formula)* Let $f : X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathscr{E}$ a locally free sheaf of finite rank. Show that there is a natural isomorphism

$$f_*(\mathcal{F} \otimes f^*\mathscr{E}) \simeq f_*(\mathcal{F}) \otimes \mathscr{E}.$$

★

11.1 *Invertible sheaves and the Picard group*

Recall that an *invertible sheaf* on a scheme X is a locally free sheaf of rank 1. We usually write L for such sheaves (they correspond to "line bundles", as we will see later). By definition, L is invertible whenever there exists a covering $\mathcal{U} = \{U_i\}$ and isomorphisms $\phi_i : \mathcal{O}_{U_i} \to L|_{U_i}$. We say that $g_i = (\phi_i)_{U_i}(1) \in L(U_i)$ is a *local generator* for L. By Lemma 11.2, a coherent $\mathcal{O}_X$-module L is invertible if and only if L_x is isomorphic to $\mathcal{O}_{X,x}$ for every $x \in X$.

Proposition 11.12 *For L, M invertible sheaves, we have*

i) *$L \otimes M$ is also an invertible sheaf. If g, h are local generators for L and M respectively, then $g \otimes h$ a local generator for $L \otimes_{\mathcal{O}_X} M$;*

ii) *$\mathscr{H}om(L, \mathcal{O}_X)$ is invertible and $\mathscr{H}om(L, \mathcal{O}_X) \otimes L \simeq \mathcal{O}_X$. If g is a local generator for L, then ψ_g defined by $\psi_g(ag) = a$ is a local generator for $\mathscr{H}om(L, \mathcal{O}_X)$;*

iii) *$\mathscr{H}om(L, M) \simeq \mathscr{H}om(L, \mathcal{O}_X) \otimes M$.*

Proof: (i) We may find a common trivialization of L and M, such that X is covered by open sets U where we have isomorphisms $\phi : \mathcal{O}_U \to L|_U$ and $\psi : \mathcal{O}_U \to M|_U$. Over such a U, we have an isomorphism $\mathcal{O}_U \simeq \mathcal{O}_U \otimes \mathcal{O}_Y \simeq L|_U \otimes M|_U$ given by $1 \mapsto 1 \otimes 1 \mapsto \phi(1) \otimes \psi(1)$ (all tensor products in this section are over $\mathcal{O}_X$. This shows (i).

For (ii), as above the fact that $\mathscr{H}om(L, \mathcal{O}_X)$ is invertible can be seen by restricting to an open where $L|_U \simeq \mathcal{O}_U$. The identity for the tensor product, and (iii) follows from Proposition 11.11. ❑

This proposition explains the term 'invertible'. Indeed, the tensor product acts as a sort of binary operation on the set of invertible sheaves; $L \otimes M$ is invertible if L and M are. Tensoring an invertible sheaf by $\mathcal{O}_X$ does nothing, so $\mathcal{O}_X$ serves as the identity. Moreover, for an invertible sheaf L we will define $L^{-1} = \mathscr{H}om(L, \mathcal{O}_X)$; by the proposition, L^{-1} is again invertible, and serves as a multiplicative inverse of $\mathcal{L}$ under $\otimes$. We can make the following definition:

$$L \otimes L^{-1} \simeq \mathcal{O}_X$$

DEFINITION 11.13 *Let X be a scheme. The Picard group* Pic(X) *is the group of isomorphism classes of invertible sheaves on X under the tensor product.*

Note that it is the set of isomorphism classes of invertible sheaves that form a group, not the invertible sheaves themselves: $L \otimes L^{-1}$ is isomorphic but strictly speaking not *equal* to $\mathcal{O}_X$.

Note also that $\operatorname{Pic}(X)$ is also an abelian group, because $L \otimes M$ is canonically isomorphic to $M \otimes L$.

Locally free sheaves on the affine line

If $\mathcal{F}$ is a coherent sheaf on $\mathbb{A}^1_k$, then $\mathcal{F} = \widetilde{M}$ for some finitely generated $k[t]$-module. The structure theorem of finitely generated modules over a PID, tells us that $M \simeq k[t]^r \oplus T$ where $r \geqslant 0$ and T is a torsion module (which is in turn a direct sum of modules of the form $k[t]/(t-a)^n$). From this we find

Proposition 11.14 *Any locally free sheaf over* $\mathbb{A}^1_k$ *is trivial;* $\operatorname{Pic}(\mathbb{A}^1_k) = 0$.