Spec A Proj R

$\mathcal{O}_X = \widetilde{A}$

$\mathcal{O}_X(D(f)) = A_f$

$\mathcal{O}_X(D_+(f)) = R_{(f)}$

$\rightsquigarrow$

$\mathcal{F} = \widetilde{M}$

M A-modul

$\widetilde{M}(D(f)) = M_f$

M gradert R-modul

$\widetilde{M}(D_+(f)) = M_{(f)}$

Chapter 12

Sheaves on projective schemes

$\left\{ \text{Graderte } R\text{-moduler} \right\} \longrightarrow \left\{ \text{Quasikoherente knipper} \right\}$

12.1 *The graded ∼-functor*

Let R be a graded ring and let GrMod_R denote the category of graded R-modules. Just like in the case of $\operatorname{Spec} A$ we will define a tilde-construction to construct sheaves on $\operatorname{Proj} R$ from graded R-modules, giving a functor GrMod_R to $\mathsf{Mod}_{\mathcal{O}_X}$. However, as we will see, this is not an equivalence of categories.

$M_{(x)} \Rightarrow (M_x)_0$

$D_+(f) \supset D_+(g)$

$\rightsquigarrow M_f \longrightarrow M_g \quad \rightsquigarrow (M_f)_0 \longrightarrow (M_g)_0$

Recall that for $f, g \in R_+$, such that $D_+(g) \subset D_+(f)$, there is a canonical localization homomorphism $\rho_{f,g} : M_{(f)} \to M_{(g)}$ where as before $M_{(f)}$ denotes the degree 0-part of the localization $\{1, f, f^2, \ldots\}^{-1}M$. It follows that we can define a $\mathscr{B}$-presheaf $\widetilde{M}$ by defining for each $D_+(f)$,

$$\widetilde{M}(D_+(f)) = M_{(f)}.$$

$(R_f)_0$

$\widetilde{M}(D_+(f)) = (M_f)_0$

$k[\frac{x_1}{x_0}]$

Note that $\widetilde{M}|_{D(f)} \simeq (\widetilde{M_{(f)}})$ via the isomorphism of $D_+(f)$ with $\operatorname{Spec} R_{(f)}$. It follows that $\widetilde{M}$ is in fact a $\mathscr{B}$-sheaf, and hence gives rise to a unique sheaf on $X = \operatorname{Proj} R$. The same identity shows that $\widetilde{M}$ is an quasi-coherent $\mathcal{O}_X$-module.

As in the Spec case, the assignment $M \mapsto \widetilde{M}$ is functorial. The following proposition summarizes the properties of this functor.

$x \leftrightarrow \mathfrak{p}$

Proposition 12.1 *The contravariant functor $M \mapsto \widetilde{M}$ has the following properties:*

- *$\sim$ is exact, commutes with direct sums and limits.*
- *The stalks satisfy $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$ for each $\mathfrak{p} \in \operatorname{Proj} R$.*
- *If R is noetherian, and M is finitely generated, then $\widetilde{M}$ is coherent.*

$(\widetilde{M})_x = (M_{\mathfrak{p}})_0$

Proving these properties is straightforward, since most of them can be checked locally on stalks. Using the isomorphisms between $D_+(f)$ and $\operatorname{Spec} R_{(f)}$ we reduce immediately to the affine case.

$D_+(x_0)$ $\qquad \widetilde{M}(D_+(x_0)) = \frac{M}{x_0^n} = 0$; $(M_{x_0})_0$

$\frac{m x_0^k}{x_0^{n+k}}$

However, it is important to note that, unlike the affine case, the functor is not faithful, as several modules can correspond to the same sheaf. For instance, take any graded R-module M such that $M_d = 0$ for all large d. Then $M_{(f)} = 0$ for all $f \in R_+$, and so $\widetilde{M} = 0$, even though M is not the 0-module. We will however see shortly that it is only the modules of this form that cause the lack of faithfulness.

$R = k[x_0, x_1]$

$M = \frac{R}{(x_0^2, x_1^2)} \rightsquigarrow \widetilde{M} = 0$

men $M \neq 0$.

$\mathbb{P}^2 \qquad x_0 \quad x_1 \quad x_2$

$$x_2 = 1 \quad \rightsquigarrow \quad u = \frac{x_0}{x_2}, \quad v = \frac{x_1}{x_2}$$

The following is useful for working with the localization of M. It says essentially that we are allowed to 'substitute in 1' when restricting a module to an affine chart $D_+(f) \subset \operatorname{Proj} R$,

Lemma 12.2 *Suppose that M is a graded R-module and $f \in R$ homogeneous of degree 1. Then there are natural isomorphisms*

$$M_{(f)} \simeq M/(f-1)M \simeq M \otimes_R R/(f-1)$$

$$(M_f)_0 \simeq \frac{M}{(f-1)M} \simeq M \otimes R/_{f-1}$$

Let us use this to compare $\widetilde{M \otimes_R N}$ with $\widetilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$. Let $f \in R$ be a homogeneous element. We have a map $M_{(f)} \times N_{(f)} \to (M \otimes_R N)_{(f)}$ sending $m/f^a \times n/f^b$ to $(m \otimes n)/f^{a+b}$. As this is $R_{(f)}$-bilinear, we get an induced map $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \to (M \otimes_R N)_{(f)}$. Since a map of $\mathscr{B}$-sheaves induces a map of sheaves, we get a natural map

$$\widetilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \to \widetilde{M \otimes_R N}. \tag{12.1}$$

Proposition 12.3 *Suppose R is generated in degree 1. Then the natural map* (12.1) *is an isomorphism.*

PROOF: By assumption, $X = \operatorname{Proj} R$ is covered by open affines of the form $D_+(f)$ where f has degree 1. For such an f, the functor $M \to M_{(f)}$ is the same as tensoring with $R/(f-1) \simeq R_{(f)}$ by the previous lemma. Furthermore,

$$(M \otimes_R R_{(f)}) \otimes_{R_{(f)}} (N \otimes_R R_{(f)}) \simeq (M \otimes_R N) \otimes_R R_{(f)}.$$

This isomorphism provides the inverse to the natural map $M_{(f)} \otimes_{R_{(f)}} N_{(f)} \to (M \otimes_R N)_{(f)}$ defined above. Then, since the map (12.1) restricts to an isomorphism on all $D_+(f)$ for $f \in R_1$, it is an isomorphism. ❑

12.2 *Serre's twisting sheaf* $\mathcal{O}(1)$

Arguably the most interesting sheaf on $X = \operatorname{Proj} R$ is the so-called *twisting sheaf*, denoted by $\mathcal{O}_X(1)$. This is a generalization of the tautological sheaf on $\mathbb{P}^n_k$, and constitutes a geometric manifestation of the fact that R is a *graded ring*.

$R(n) := R$ (som modul)

grading: $(R(n))_d = R_{d+n}$

For an integer n, we will define an R-module $R(n)$ as follows: As an underlying module $R(n)$ is just R, but with the grading shifted by n:

$$R(n)_d = R_{d+n}$$

This is naturally a graded R-module and hence gives rise to a quasi-coherent $\mathcal{O}_X$-module on X.

$R = k[x_0, x_1] \rightsquigarrow (R(1))_0 = kx_0 \oplus kx_1$

$R(1)$ blir en modul over R.

$\mathcal{F}$ invertibel: · $\mathcal{F}$ lokalt fri rang 1.
· $\mathcal{F}|_U \simeq \mathcal{O}_U \quad U \in \mathcal{U}$.

$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

Definition 12.4 *For an integer n, we define*

$$\mathcal{O}_X(n) = \widetilde{R(n)}.$$

For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X, we define the twist by n *by* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

$R = k[x_0, x_1]$

$x_0^2 \rightsquigarrow$ sekcjon i $\Gamma(X, \mathcal{O}_X(2))$

For an element $r \in R_d$, there is a corresponding section in $\Gamma(X, \mathcal{O}_X(d))$. This is because we can think of an element of $\Gamma(X, \mathcal{O}_X(d))$ as a collection of elements $(r_f, D_+(f))$ with $r_f \in (R_f)_d$ matching on the overlaps $D_+(f) \cap D_+(g)$. Hence we can define an R_0-module homomorphism

$$R_d \to \Gamma(X, \mathcal{O}_X(d))$$

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X(d)) \to \prod_{f \in R_+} \Gamma(D_+(f), \mathcal{O}(d)) \to \prod_{f,g} \Gamma \dots$$

$X = \operatorname{Spec} k[x]$ $\quad x \rightsquigarrow$ global sekcjon i $\mathcal{O}_X$

by $r \mapsto (r/1, D_+(f))$. (On the overlaps $D_+(fg)$ it is clear that the two elements $(r/1, D_+(f))$ and $(r/1, D_+(g))$ become equal, so this defines an actual global section of $\mathcal{O}(d)$). Abusing notation, we will also denote the section by r.

$$(R(n)_f)_0 = f^n \cdot (R_f)_0$$

$$\frac{r f^n}{f^{N+n}} = \frac{r}{f^N} \qquad \deg r - N = n$$

Line bundles on Proj R defined over
on $D_+(f)$
$f \in R_1$

Note that if $f \in R_1$, then $R(n)_{(f)} = f^n R_{(f)}$. Thus, on the affine $D_+(f)$, we have $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_X|_{D_+(f)}$. In particular, $\mathcal{O}_X(n)|_{D_+(f)} \simeq \mathcal{O}_{D_+(f)}$. In other, words $\mathcal{O}_X(n)$ is a locally free sheaf of rank 1, that is, an invertible sheaf. By the

$$\deg r - (N + n) = 0$$

$$\frac{r}{f^N} = f^n \cdot r'$$

$$r' \in (R_f)_0 \qquad r' = \frac{r}{f^{N+n}}$$

Proposition 12.5 *If R is generated in degree 1, then $\mathcal{O}_X(n)$ is an invertible sheaf for every n. Moreover, there is a canonical isomorphism*

$$\mathcal{O}_X(m+n) \simeq \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

Indeed, if R is generated in degree 1, Proposition 12.3 shows that $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n)$ is the sheaf associated to $R(m) \otimes_R R(n) \simeq R(n+m)$, i.e., $\mathcal{O}_X(n+m)$.

Example 12.6 Consider again the example of projective space $\mathbb{P}_k^n = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$. We use the covering $\{U_i\}$ of the distinguished open sets $U_i = D(x_i) \simeq \operatorname{Spec}\left(k[x_0, \ldots, x_n]_{(x_i)}\right) = \operatorname{Spec}\left(k[x_0 x_i{}^{-1}, \ldots, x_n x_i{}^{-1}]\right)$. Then $R(l)_{(x_i)} = \left(k[x_0, \ldots, x_n]_{(x_i)}\right)_l = x_i^l k[x_0 x_i{}^{-1}, \ldots, x_n x_i{}^{-1}]$, and so

$$\Gamma\left(U_i, \mathcal{O}(l)\right) = x_i^l k[x_0 x_i{}^{-1}, \ldots, x_n x_i{}^{-1}]$$

$D_+(x_0 x_1) = \operatorname{Spec} R_{(x_0 x_1)}$

$D_+(x_i)$

On the overlaps, $\left(R(l)_{x_i}\right)_{\frac{x_j}{x_i}} = R(l)_{x_i x_j} = \left(R(l)_{x_j}\right)_{\frac{x_i}{x_j}}$ we find that two regular sections

$$x_i^l s_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

and

$$x_j^l s_j\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right)$$

restrict to the same section of $\mathcal{O}(l)|_{U_i \cap U_j}$ if and only if

$$s_j\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) = \left(\frac{x_i}{x_j}\right)^l s_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

Serre:

$$\Gamma(X, \mathcal{O}(d)) = \left\{ \begin{array}{l} \text{homogène polynôme} \\ \text{i } x_0 \ldots x_n \text{ au} \\ \text{grad } d \end{array} \right\}$$

Mål: $\mathcal{F}$ qc på $X = \operatorname{Proj} R$

$\rightsquigarrow$ $\mathcal{F} = \widetilde{M}$ for en gradert R-modul M.

12.3 Extending sections of invertible sheaves

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $x \in X$ be a point. We define the *fiber of* $\mathcal{F}$ *at* x to be the $k(x)$-vector space

$$\mathcal{F}(x) : \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$$

$x \in U$ $\quad \mathcal{F}(x) = \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x \simeq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$

$s \in \mathcal{F}(U) \rightsquigarrow s(x) \in \mathcal{F}(x)$.

If U is an open set containing x and $s \in \Gamma(U, \mathcal{F})$, we denote by $s(x)$ the image of the germ $s_x \in \mathcal{F}_x$ in $\mathcal{F}(x)$.

DEFINITION 12.7 *Let L be an invertible sheaf and $f \in \Gamma(X, L)$. We define the open set X_f as*

$$X_f = \{x \in X | f(x) \neq 0\}.$$

Equivalently, X_f is the set of points where $f \notin \mathfrak{m}_x L_x$.

$\therefore$ $V(f) = \{x \in X \mid f(x) = 0\}$ er lukket.

X_f is indeed an open set of X: L is locally free, so through every point there is a neighbourhood U such that $L|_U \simeq \mathcal{O}_X|_U$. To show that it is open, we can therefore assume that $L = \mathcal{O}_X$. If $x \in X_f$ there exists a $t_x \in \mathcal{O}_{X,x}$ such that $f_x t_x = 1$. Choose an open neighbourhood V of x such t_x is represented by a section $t \in \Gamma(V, \mathcal{O}_X)$. By shrinking V we can assume that $(f|_V)t = 1$ on V, and so $V \subset X_f$ is open.

$s|_{D(f)} \Rightarrow f^n \cdot s = 0 \quad n \gg 0$

Lemma 12.8 *Suppose X is a noetherian scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and $\mathcal{L}$ an invertible sheaf. Suppose $f \in \Gamma(X, \mathcal{L})$. Then:*

- *i) If a section $t \in \Gamma(X, \mathcal{F})$ restricts to zero on X_f, then there is an integer N such that $t \otimes f^N \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^N)$ is zero (on all of X).*
- *ii) Suppose $t \in \Gamma(X_f, \mathcal{F})$. Then there is an integer N such that $t \otimes f^N$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^N$.*

$\frac{1}{X^2} \in \Gamma(D_+(x), \mathcal{O})$

$\rightsquigarrow x^2 \otimes \frac{1}{x^2}$ extends.

$\mathcal{F}$ q.c. $\overset{?}{\rightsquigarrow}$ $\mathcal{F} = \widetilde{M}$

Hvilken R-modul ?

12.4 *The associated graded module*

We have associated to a graded R-module M a sheaf $\widetilde{M}$ on $X = \operatorname{Proj} R$. To classify quasi-coherent sheaves on X we would, like in the case of affine schemes, give some sort of inverse to this assignment. However, unlike the case for $X = \operatorname{Spec} A$, we can not simply take the global sections functor. Indeed, even for $\mathcal{F} = \mathcal{O}_X$ on $X = \mathbb{P}^1_k$, $\Gamma(X, \mathcal{O}_X) = k$, from which we certainly cannot recover $\mathcal{F}$. The remedy is to look at the various Serre twists $\mathcal{F}(m)$ – in fact all of them at once:

$$\Gamma_*(\mathcal{O}_X) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{O}(n))$$

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(n)$$

DEFINITION 12.10 *Let R be a graded ring and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X = \operatorname{Proj} R$. We define the* graded R-module associated *to $\mathcal{F}$, $\Gamma_*(\mathcal{F})$ as*

(M=)

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

R-modul

$r \in R_d \rightsquigarrow$ seksjon $r \in \Gamma(X, \mathcal{O}(d))$

$$\mathcal{F}(n) \otimes \mathcal{O}(d) \longrightarrow \mathcal{F}(n+d)$$

This has the structure of an R-module: If $r \in R_d$, we have a corresponding section $r \in \Gamma(X, \mathcal{O}_X(d))$ (abusing notation, as before). So if $\sigma \in \Gamma(X, \mathcal{F}(n))$ then we define $r \cdot \sigma \in \Gamma(X, \mathcal{F}(n+d))$ by the tensor product $r \otimes \sigma$, and using the isomorphism $\mathcal{F}(n) \otimes \mathcal{O}(d) \simeq \mathcal{F}(n+d)$. In particular, $\Gamma_*(\mathcal{O}_X)$ is a graded *ring*.

Exercise 12.1 Let k be a field and let $R = k[x_0, \ldots, x_n]$. Let $\pi : \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n_k = \operatorname{Proj} R$ denote the 'quotient morphism' from Exercise 9.9. Show that for a graded R-module M, we have

$$\pi_*(\widetilde{M}|_{\mathbb{A}^{n+1}_k - 0}) = \bigoplus_{n \in \mathbb{Z}} \widetilde{M}(d)$$

Sections of the structure sheaf

Proposition 12.11 *Let R be a graded integral domain, finitely generated in degree 1 by elements $x_0, \dots, x_n$, and let $X = \operatorname{Proj} R$. Then*

i) $\Gamma_*(\mathcal{O}_X) = \bigcap_{i=0}^{n} R_{(x_i)} \subset K(R)$

ii) If each x_i is a prime element, then $R = \Gamma_*(\mathcal{O}_X)$.

PROOF: Cover X by the opens $U_i = D_+(x_i)$. We have, since $\Gamma(D_+(x_i), \mathcal{O}(m)) \simeq (R_{x_i})_m$, that the sheaf axiom sequence takes the following form

$\Gamma(X, \mathcal{O}_X(m))$

$$0 \to \mathcal{O}(m) \to \bigoplus_{i=0}^{n} (R_{x_i})_m \to \bigoplus_{i,j} (R_{x_i x_j})_m$$

Taking directs sums over all m, we get

$$0 \to \Gamma_*(\mathcal{O}_X) \to \bigoplus_{i=0}^{n} (R_{x_i}) \to \bigoplus_{i,j} (R_{x_i x_j})$$

So a section of $\Gamma_*(\mathcal{O}_X)$ corresponds to an $(n+1)$-tuple $(t_0, \ldots, t_n) \in \bigoplus_{i=0}^{n}(R_{x_i})$ such that t_i and t_j coincide in $R_{x_i x_j}$ for each $i \neq j$. Now, the x_i are not zero-divisors in R, so the localization maps $R \to R_{x_i}$ are injective. It follows that we can view all the localizations R_{x_i} as subrings of $R_{x_0 \ldots x_n}$, and then $\Gamma_*(\mathcal{O}_X)$ coincides with the intersection

$$\bigcap_{i=0}^{n} R_{x_i} \subset A[x_0^{\pm 1}, \cdots, x_n^{\pm 1}].$$

In the case each x_i is prime, this intersection is just R. ❑

$\mathbb{P}^1, \ \mathcal{O}(1)$

$$\Gamma(\mathbb{P}^1, \mathcal{O}(1)) \to k\left[\frac{x_1}{x_0}\right] \cdot x_0 \oplus k\left[\frac{x_0}{x_1}\right] \cdot x_1 \to k\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right] x_0 = k\left[\frac{x_0}{x_1}, \frac{x_1}{x_0}\right] x_1$$

$$s\left(\frac{x_1}{x_0}\right) x_0 = t\left(\frac{x_0}{x_1}\right) \cdot x_1$$

Corollary 12.12 *Let* $X = \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$ *for a ring* A. *Then*

$$\Gamma_*(\mathcal{O}_X) \simeq A[x_0, \ldots, x_n]$$

In particular we can identify $\Gamma(\mathbb{P}^n_A, \mathcal{O}(d))$ *with the A-module generated by homogeneous degree d polynomials.*

When R is not a polynomial ring, it can easily happen that $\Gamma_*(\mathcal{O}_X)$ is different than R. Here is a concrete example:

EXAMPLE 12.13 *(A quartic rational curve)* Let k be a field and let R be the k-algebra $R = k[s^4, s^3t, st^3, t^4] \subset k[s,t]$. Note that the monomial s^2t^2 is missing from the generators of R. Define the grading such that $R_1 = k \cdot \{s^4, s^3t, st^3, t^4\}$.

$x_0 = s^4$

$x_1 = s^3 t$

$x_2 = s t^3$, $x_3 = t^4$

We can also think of R as the graded ring

$$R = k[x_0, x_1, x_2, x_3]/(x_0^2 x_2 - x_1^3, x_1 x_3^2 - x_2^3, x_0 x_3 - x_1 x_2).$$

We have a covering $\operatorname{Proj} R = U_0 \cup U_1$, where

$$U_0 = \operatorname{Spec}(R_{(x_0)}) \text{ and } U_1 = \operatorname{Spec}(R_{(x_3)}).$$

Here $R_{(x_0)} = k[\frac{t}{s}, \frac{t^3}{s^3}, \frac{t^4}{s^4}] = k[\frac{t}{s}]$ and $R_{(x_3)} = k[\frac{s}{t}]$. So $\operatorname{Proj} R$ is in fact isomorphic to $\mathbb{P}^1$. We have shown that X embeds as a rational (degree 4) curve in $\mathbb{P}^3$.

$k[x_0, x_1]$

$X = \mathbb{P}^1$ $\Gamma(X, \mathcal{O}(1)) = k x_0 \oplus k x_1$

What is $\Gamma(X, \mathcal{O}_X(1))$? On the opens we find $\mathcal{O}_X(1)(U_0) = k\left[\frac{t}{s}\right] \cdot s^4$ and $\mathcal{O}_X(1)(U_1) = k\left[\frac{s}{t}\right] \cdot t^4$. So using the sheaf sequence, we get

$$0 \to \Gamma(X, \mathcal{O}_X(1)) \to k\left[\frac{s}{t}\right] s^4 \oplus k\left[\frac{t}{s}\right] t^4 \to k\left[\frac{s}{t}, \frac{t}{s}\right] u^4$$

Note that the monomial s^2t^2 belongs to both $k\left[\frac{s}{t}\right]t^4$ and $k\left[\frac{t}{s}\right]s^4$, and so defines an element in $\Gamma(X, \mathcal{O}_X(1))$. In fact,

$$\Gamma(X, \mathcal{O}_X(1)) = k\{s^4, s^3t, s^2t^2, st^3, t^4\}$$

even though $R_1 = k\{s^4, s^3t, st^3, t^4\}$.

Thm $\bar{R} = \Gamma_*(\mathcal{O}_X)$.

In this example, the graded ring $\Gamma_*(\mathcal{O}_X) = k[s^4, s^3t, st^3, t^4]$ is the integral closure of R. We will see later that this is not a coincidence. ☆

The homomorphism α.

Let $X = \operatorname{Proj} R$, where R be a graded ring and let M be a graded R-module. It is a natural question how to recover M from the sheaf $\widetilde{M}$. We will define a homomorphism of graded R-modules called *the saturation map*

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

As before, it is useful to think of elements in $\Gamma(X, \widetilde{M}(n))$ as a collection of elements $(m_f, D_+(f))$ for $m \in (M_{(f)})_n$ and $f \in R$ matching on the various overlaps.

Proposition 12.14 *When R is generated in degree 1, there is a graded R-module homomorphism*

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

Indeed, we can define α by sending an element $m \in M_d$ to the collection given by $(m/1, D_+(f))$, where f ranges over R_1. On the overlaps $D_+(f) \cap D_+(g) = D_+(fg)$ it is clear that the two elements $(m/1, D_+(f))$ and $(m/1, D_+(g))$ become equal so this defines an actual global section of $\widetilde{M}(n)$. We see that this is a graded homomorphism. Moreover, it is functorial in M.

Lemma 12.15 *If R is a graded Noetherian integral domain generated in degree one. Then $R' = \Gamma_*(\mathcal{O}_X)$ is an integral extension of R.*

PROOF: Let $x_1, \ldots, x_r$ be degree one generators of R. Let $\alpha : R \to R' = \Gamma_*(\mathcal{O}_X)$, be the map above. It is clear that the map is injective: If $r \in R$ is an element so that $r/1 = 0$ over every $R_{(f)}$, then $r = 0$.

To show integrality, let $s \in R'$ be a homogeneous element of non-negative degree. By quasi-compactness, we can find an $n > 0$, so that $\alpha(x_i^n)s \in \alpha(R)$ for every i. R_m is generated by monomials in x_i of degree m, so $\alpha(R_m)s \subset \alpha(R)$ for m large (e.g., $m \geqslant kn$). Let $R^{\geqslant kn}$ be the ideal of R generated by elements of degree $\geqslant kn$. We have that $\alpha(R^{\geqslant kn})s \subset \alpha(R^{\geqslant kn})$. Moreover, since R is noetherian, $R^{\geqslant rn}$ is finitely generated, so applying the Cayley–Hamilton theorem, we get that s satisfies an integral equation over R. Hence R' is integral over R. ❑

Mail: $\mathcal{F}$ qc pi Proj R

$\Rightarrow \mathcal{F} = \widetilde{M}$ $M = \Gamma_*(X, \mathcal{F})$.

The map β

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We get a natural R-module $\Gamma_*(\mathcal{F})$, and in turn a sheaf of $\mathcal{O}_X$-modules $\widetilde{\Gamma_*(\mathcal{F})}$. We will define a map of $\mathcal{O}_X$-modules

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F} \tag{12.2}$$

$$\Gamma(D_+(f), \tilde{M}) \ni \frac{m}{f^d}$$

$$M = \Gamma_*(\mathcal{F}) \qquad f^{-d} \in \Gamma(D_+(f), \mathcal{O}_X(-d))$$

as follows. Let $f \in R_1$. We will define β over $D_+(f)$. A section of $\widetilde{\Gamma_*(\mathcal{F})}$ is represented on $D_+(f)$ by a fraction m/f^d where $m \in \Gamma(X, \mathcal{F}(d))$. If we think of f^{-d} as a section in $\mathcal{O}(-d)(D_+(f))$, then we can consider the tensor product $m \otimes f^{-d}$ which is a section of $\mathcal{F}$ via the isomorphism $\mathcal{F}(d) \otimes \mathcal{O}(-d) \simeq \mathcal{F}$. This is compatible with the module structures, so we obtain a homomorphism of $\mathcal{O}_X$-modules

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$$

by associating m/f^d to $m \otimes f^{-d}$.

Proposition 12.16 *Suppose R is a graded ring, finitely generated in degree 1 over R_0. Suppose $\mathcal{F}$ is a quasi-coherent sheaf on* $\operatorname{Proj} R$*. Then the map*

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F} \tag{12.3}$$

is an isomorphism.

PROOF: Since R is generated by R_1 over R_0, the open sets $D_+(f)$ with $f \in R_1$ cover X. To show that (12.3) is an isomorphism of sheaves, it is sufficient to prove it on such an open.

$s \in \Gamma(D_+(f), \mathcal{F})$

$t = s \otimes f^N$ utvider for $N >> 0$

Let $f \in R_1$, and consider it as a section of $\Gamma(X, \mathcal{O}(1))$. Then taking $L = \mathcal{O}(1)$ in Lemma 12.8, point (i) there says that if an element s of $\Gamma(D_+(f), \mathcal{F})$ is given, we can find some element t of $\Gamma_*(\mathcal{F})_N$ (for N sufficiently large) such that $t \otimes f^{-N} \in \Gamma(D_+(f), \mathcal{F})$ equals s. This implies that the map β is surjective.

$s = t \cdot f^{-N}$

$\Rightarrow$ s er i bildet av β.

$$\widetilde{\Gamma_*(\mathcal{F})} \xrightarrow{\beta} \mathcal{F}$$

For injectivity, suppose $s \in \Gamma(X, \mathcal{F}(n))$ is such that $s \otimes f^{-n} = 0$ on $D_+(f)$, i.e. $s/f^n \in \Gamma_*(\mathcal{F})_{(f)}$ is in the kernel of (12.3) on the $D_+(f)$-sections. Then the lemma implies that there is a power f^N with $s \otimes f^N \in \Gamma(X, \mathcal{F}(n+N)) = 0$. This states that $s/f^n = 0$ in $\Gamma_*(\mathcal{F})_{(f)}$ by the definition of localization and so the map is injective. ❑

We have now defined two functors

$$\sim: \mathsf{GrMod}_R \to \mathsf{QCoh}_X$$

$$M \longrightarrow \widetilde{M}$$

← ikke alltid injektiv

and

$$\Gamma_* : \mathsf{QCoh}_X \to \mathsf{GrMod}_R$$

$$\mathcal{F} \longmapsto \Gamma_*(\mathcal{F})$$

$$M = k[x_0, x_1] / (x_0^2, x_1^2) \rightsquigarrow \widetilde{M} = 0$$

Since $\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is an isomorphism, it follows that $\sim$ is essentially surjective. However, unlike the affine case, the functors do not give mutual inverses. This is because, as we have seen, that $\sim$ is not faithful; the $\sim$ of any module M which is finite over R_0 is the zero sheaf.

We can define an equivalence relation on graded modules by setting $M \sim N$ if $\bigoplus_{i \geqslant i_0} M_i \simeq \bigoplus_{i \geqslant i_0} N_i$ for some $i_0 \in \mathbb{Z}$. For two finitely generated graded R-modules M, N we have $M \sim N$ if and only if $\tilde{M} \simeq \tilde{N}$, so we have identified precisely the 'kernel' of the functor $\sim$.

Putting everything together, we find

Theorem 12.17 *Let R be a graded ring, finitely generated in degree 1 over R_0 and let $X = \operatorname{Proj} R$. Then the functors*

$$\sim\colon \mathsf{GrMod}_R \to \mathsf{QCoh}_X$$

and

$$\Gamma_* : \mathsf{QCoh}_X \to \mathsf{GrMod}_R^{sat}$$

satisfy $\widetilde{\Gamma_*(\mathcal{F})} = \mathcal{F}$ *for all* $\mathcal{F} \in \mathsf{QCoh}_X$*, and give an equivalence between the categories of quasi cohoherent sheaves on* X *and graded* R*-modules modulo the equivalence relation* $M \sim N$*.*

$Y \xrightarrow{i} X \iff$ quasicoherent ideal-sheaf $\mathcal{I}$

$$(Y, \mathcal{O}_Y) = \left(\operatorname{supp}\left(\frac{\mathcal{O}_X}{\mathcal{I}}\right), \frac{\mathcal{O}_X}{\mathcal{I}}\right)$$

12.5 *Closed subschemes of* Proj R

Having discussed what quasi-coherent sheaves are on projective spectra, we will now use this to study closed subschemes. We saw earlier that given a graded ideal $I \subset R$ we could associate a closed subscheme $V(I) \subset \operatorname{Proj} R$ and a closed immersion $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$. On the other hand, we also saw above that immersion $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$. On the other hand, we also saw above that many graded modules M could give rise to the same quasi-coherent sheaf $\widetilde{M}$. This is also the case for graded ideals , as we shall see, but luckily we are again able to completely identify which ideals give rise to the same closed subscheme.

$I \subset R$ homogeneous ideal

$B = R_+$

In the discussion it will be convenient to introduce the *saturation* of an ideal. The upshot will be that this will serve as the 'largest' ideal corresponding to a given subscheme. We fix an ideal $B \subset R$ (which will typically be the irrelevant ideal R_+). Then for a graded ideal $I \subset R$, we define the *saturation* of I with respect to an ideal B is defined as the ideal

$$I : B^{\infty} := \bigcup_{i \geqslant 0} I : B^{i} = \{r \in R | B^{n} r \in I \text{ for some } n \geqslant 0\}.$$

Note: $I \subseteq I : B$

We say that I is B-saturated if $I = I : B^{\infty}$ and more concisely, *saturated* if it is R_+-saturated. We will here denote $I : (R_+)^{\infty}$ by $\overline{I}$. Note that the ideal $\overline{I}$ is homogeneous if I is.

$$(x_0^2, x_0x_1) = (x_0) \cap (x_0^2, x_1)$$

$$I = (x_0^2, x_0x_1) \rightsquigarrow I : (x_0, x_1) = (x_0)$$

Example 12.18 In $R = k[x_0, x_1]$, the (x_0, x_1)-saturation of (x_0^2, x_0x_1) is the ideal (x_0). Note that both (x_0) and (x_0^2, x_0x_1) define the same subscheme of $\mathbb{P}^1_k$, but in some sense the latter ideal is inferior, since it has a component in the irrelevant ideal (x_0, x_1). This example is typical; the saturation is a process which throws away components of I supported in the irrelevant ideal. ☆

$\mathbb{P}^n_A$

PROPOSITION 12.19 *Let A be a ring and let $R = A[x_0, \ldots, x_n]$.*

i) To each closed subscheme Y of $\mathbb{P}^n_A$, there is a corresponding homogeneous saturated ideal $I \subset R$, such that Y corresponds to the subscheme $\operatorname{Proj}(R/I) \to \operatorname{Proj} R$.

ii) Two ideals I, J defined the same subscheme if and only if they have the same saturation.

$\overline{I} = \overline{J}$ $\quad I : B^\infty = J : B^\infty \quad B = (x_0 \ldots x_n)$

iii) If $Y \subset \mathbb{P}^n_A$ is a closed subscheme with ideal sheaf $\mathcal{I}$, then $\Gamma_*(\mathcal{I})$ is a saturated ideal of R. In fact, the ideal $\Gamma_*(\mathcal{I})$ is the largest ideal that defines the subscheme Y.

"kanoniske" idealet som definerer Y.

In particular, there is a 1-1 correspondence between closed subschemes $i : Y \to \mathbb{P}^n_A$ and saturated homogeneous ideals $I \subset R$.

PROOF: (i) Let $i : Y \to \mathbb{P}^n_A$ be a subscheme of $\mathbb{P}^n_A = \operatorname{Proj} R$ and let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^n_A}$ denote the ideal sheaf of Y. Using the fact that global sections is left-exact, we have $\Gamma_*(\mathcal{I}) \subset \Gamma_*(\mathcal{O}_{\mathbb{P}^n_A}) = R$. $I = \Gamma_*(\mathcal{I})$ is naturally an R-module, so in fact I is a (homogeneous) ideal of R.

Any such ideal I gives rise to a closed subscheme $i' : \mathrm{Proj}(R/I) \to \mathbb{P}^n_A$ and hence an ideal sheaf $\mathcal{J}$ satisfying $\widetilde{I} = \mathcal{J}$. By Proposition 12.16, we also have $\widetilde{I} = \mathcal{I}$, so the two quasi-coherent ideal sheaves coincide and i is indeed the same as i'. By construction $I = \Gamma_*(\tilde{I})$, so I is saturated.

(ii) If I, J define the same subscheme, they have the same ideal sheaf $\tilde{I} = \mathcal{I} = \mathcal{J} = \tilde{J}$ on $\mathbb{P}^n_A$. Let $r \in I_d$, then on $U_i = D_+(x_i)$, the fraction rx_i^{-d} defines an element of $\Gamma(U_i, \widetilde{I}) = \Gamma(U_i, \widetilde{J})$. Since also $\mathcal{I}$ corresponds to J, we have $rx_i^{-d} = t_i x_i^{-d}$ for some $t_i \in J_d$ of degree d. Hence there is a power n_i such that $x_i^{n_i}(r - t_i) = 0$ in R. This shows that r is in the saturation of J. By symmetry, we have $\overline{I} = \overline{J}$.

(iii) Let $r \in R$ be such that $x_i^{n_i} r \in \Gamma_*(\mathcal{I})$ (that is $r \in \overline{\Gamma_*(\mathcal{I})}$). Let $m = \max n_i$. We want to show that $r \in \Gamma_*(\mathcal{I})_d = \Gamma(X, \mathcal{I}(d))$. On $U_i = D_+(x_i)$, we see that $x_i^{-m} \otimes x_i^m r$ defines a section of $\mathcal{I}(d+m) \otimes \mathcal{O}(-n)$. The latter is isomorphic to $\mathcal{I}(d)$ and $x_i^{-m} \otimes x_i^m r = r$ via this isomorphism. So $r \in \Gamma(U_i, \mathcal{I}(d))$. Hence $r \in \Gamma(X, \mathcal{I}(d)) \subset \Gamma_*(\mathcal{I})$, and $\Gamma_*(\mathcal{I})$ is saturated. ❑

Veronese ringen $R^{(n)}$

Example 12.20 Let k be a field and let $R = k[u, v]$. Moreover introduce the graded ring $S = R^{(n)} = k[u^n, u^{n-1}v, \ldots, v^n]$. We have a graded surjection

$$\phi : k[x_0, \ldots, x_n] \to S$$

given by $x_i \mapsto u^i v^{n-i}$ for $i = 0, \ldots, n$. The ideal $I = \operatorname{Ker} \phi$ is generalted by the 2×2-minors of the matrix

$\mathbb{P}^1_k \xrightarrow{\ i\ } \mathbb{P}^n_k$ rational normal curve

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$$

Thus we have an embedding of $\mathbb{P}^1_k = \operatorname{Proj} S$ into $\mathbb{P}^n$ with image $V(I)$. The image is called a *rational normal curve of degree* n. Note that for $n = 2$, the image of $\mathbb{P}^1_k \to \mathbb{P}^2_k$ is the conic $x_1^2 = x_0 x_2$. ☆

12.6 *The Segre embedding*

Recall that for affine schemes $X = \operatorname{Spec} B, Y = \operatorname{Spec} C$ over $S = \operatorname{Spec} A$, the fiber product $X \times_S Y$ was defined as $\operatorname{Spec}(B \otimes_A C)$. There is a similar statement for Proj:

$$S = \bigoplus_{n \geq 0} (R_n \otimes_A R'_n)$$

← graded ring

Theorem 12.21 *Let R, R' be graded rings with $R_0 = R'_0 = A$. Let $S = \bigoplus_{n \geqslant 0}(R_n \otimes R'_n)$. Then*

$$\operatorname{Proj} S \simeq \operatorname{Proj} R \times_A \operatorname{Proj} R'$$

Corollary 12.22 *Let A be a ring and let $m, n \geqslant 1$ be integers. Then there is a closed immersion*

$$\sigma_{m,n} : \mathbb{P}^m_A \times_A \mathbb{P}^n_A \to \mathbb{P}^{mn+m+n}_A$$

Proof: Consider the A-algebra $S = \bigoplus_{n \geqslant 0}(R_n \otimes R'_n)$ above, where $R = A[x_0, \ldots, x_m]$ and $R' = A[y_0, \ldots, y_n]$ are the polynomial rings. Consider the following morphism of graded A-algebras.

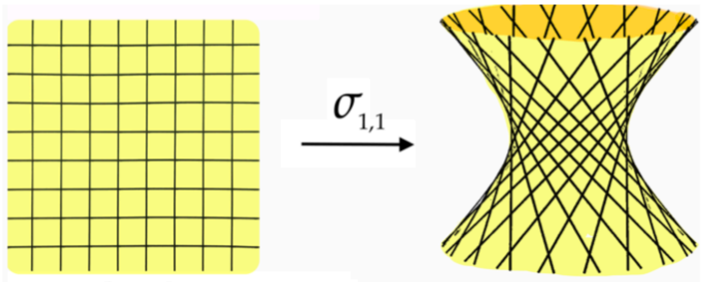
$$\begin{aligned} A[z_{ij}]_{0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n} &\to A[x_0, \ldots, x_m] \otimes A[y_0, \ldots, y_n] \\ z_{ij} &\mapsto x_i \otimes y_j. \end{aligned}$$

It is clear that S is generated as an $R_0 \otimes R'_0$-algebra by the products $x_i \otimes y_j$, so the map is surjective and thus we get the desired closed immersion. ❑

Example 12.23 Let $R = k[x_0, x_1]$, $R' = k[y_0, y_1]$. Then $u_{ij} = x_i y_i$ defines an isomorphism

$$S = \bigoplus_{n \geqslant 0} (R_n \otimes R'_n) \to k[u_{00}, u_{01}, u_{10}, u_{11}]/(u_{00}u_{11} - u_{01}u_{10})$$

This recovers the usual embedding of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ as a quadric surface in $\mathbb{P}^3_k$.



A smooth quadric surface

12.7 *Two important exact sequences*

Hypersurfaces

Let $R = k[x_0, \ldots, x_n]$ and $\mathbb{P}^n_k = \operatorname{Proj} R$. Let $F \in R$ denote a homogeneous polynomial of degree $d > 0$. F determines a projective hypersurface $X = V(F)$,

$$X = \operatorname{Proj}\left(R/(F) \right) \hookrightarrow \mathbb{P}^n_k$$

which has dimension $n-1$. Note that $I(X) = (F)$ We then have an isomorphism

$$R(-d) \to I(X)$$

given by multiplication with F. Note the shift here: The constant '1' gets sent to F should have degree d on both sides! This gives the sequence of R-modules

$$0 \to R(-d) \to R \to R/(F) \to 0$$

sjekk dette på hver $D_+(f)$.

We have $\widetilde{R(-d)} = \mathcal{O}_{\mathbb{P}^n_k}(-d)$ and $\widetilde{(R/F)} = i_*\mathcal{O}_X$, where $i : X \to \mathbb{P}^n_k$ is the inclusion, so we get the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \to \mathcal{O}_{\mathbb{P}^n_k} \to i_*\mathcal{O}_X \to 0$$

brukes mye til å regne ut invarianter til X (grad, genus, ...)

⟶ skal se på dette i Kap. 15.

dimension $\dim X = n-2$

Complete intersections

Let F, G be two homogeneous polynomials without common factors of degrees d, e respectively. Let $I = (F, G)$ and $X = V(I) \subset \mathbb{P}^n_k$. X is called a 'complete intersection' – it is the intersection of the two hypersurfaces $V(F)$ and $V(G)$. To study X we have exact sequences

$$0 \to R(-d-e) \xrightarrow{\alpha} R(-d) \oplus R(-e) \xrightarrow{\beta} I \to 0$$

$e_0 \qquad e_1 \qquad e_2$

$e_0 \overset{\alpha}{\longmapsto} F \cdot e_2 - G \cdot e_1$

$e_1 \longrightarrow F$

$e_2 \longrightarrow G$

The maps here are defined by $\alpha(h) = (-hG, hF)$ and $\beta(h_1, h_2) = h_1 F + h_2 G$. These maps preserve the grading.

To prove exactness, we start by noting that α is injective (since R is an integral domain) and β is surjective (by the defintion of I). Then if $(h_1, h_2) \in \operatorname{Ker} \beta$, we have $h_1 F = -h_2 G$, which by the coprimality of F, G means that there is an element h so that $h_1 = -hG$, $h_2 = hF$.

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_X \to 0$$

Applying $\sim$, we obtain the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_X \to 0$$

These sequences are fundamental in computing the geometric invariants from X. We will see several examples of this later.

12.8 *Two examples of locally free sheaves*

Projective space

Let k be a field and write $\mathbb{P}^n = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$. Consider the map of graded modules $\phi : R(-1) \to R^{n+1}$ sending $1 \in R$ to the element $(x_0, \ldots, x_n) \in R^{n+1}$. This map is clearly injective, so we get an exact sequence

$$0 \to R(-1) \xrightarrow{\phi} R^{n+1} \to M \to 0$$

$$1 \longmapsto (x_0, x_1, \ldots, x_n)$$

where $M = \operatorname{Coker} \phi$. Applying $\sim$, we get an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}_{\mathbb{P}^n_k}^{n+1} \to \mathscr{E} \to 0 \tag{12.4}$$

where $\mathscr{E} = \widetilde{M}$. We claim that $\mathscr{E}$ is locally free of rank n. Indeed, on the

$$0 \to \left(R_{x_0}(-1)\right)_0 \xrightarrow{\phi} \left(R_{x_0}\right)_0^{n+1} \to \left(M_{x_0}\right)_0 \to 0$$

$\curvearrowleft$ x_0 invertible?

distinguished open set $D_+(x_0) = \operatorname{Spec} R_{(0)}$, we have

$$\begin{aligned}
\mathscr{E}(D_+(x_0)) &= \left(\bigoplus_{i=0}^{n} R/(x_0e_0 + \cdots + x_ne_n)\right)_{(x_0)} \\
&= \left(\bigoplus_{i=0}^{n} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]e_i\right) / (e_0 + \frac{x_1}{x_0}e_0 \cdots + \frac{x_n}{x_0}e_n) \\
&\simeq \bigoplus_{i=1}^{n} k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]e_i
\end{aligned}$$

Hence $\mathscr{E}|_{D_+(x_0} \simeq \mathcal{O}_{U_0}$. By a symmetric argument, $\mathscr{E}$ is free also on the other $D_+(x_i)$, so it is locally free of rank n. We will show in Section 15.6 that $\mathscr{E}$ is not free, and in fact not even isomorphic to a direct sum of invertible sheaves.

The four-dimensional quadric hypersurface

Let k be a field and let $R = k[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}]$. Consider the matrix

$$M = \begin{pmatrix} p_{12} & p_{13} & p_{23} & 0 \\ -p_{02} & -p_{03} & 0 & p_{23} \\ p_{01} & 0 & -p_{03} & -p_{13} \\ 0 & p_{01} & p_{02} & p_{12} \end{pmatrix}$$

$: R^4 \longrightarrow R^4$

Let us consider the loci in $\mathbb{P}^5 = \operatorname{Proj} R$ where this matrix has a given rank. Note that M has rank $\leqslant 3$ precisely when the determinant vanishes. In fact, this matrix M has the special property that the determinant is a square: $\det M = q^2$ where

$$q = p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$$

The locus of points where M has rank 2 is given by the ideal generated by the 2×2-minors, which by direct calculation has radical equal to the irrelevant ideal R_+. Consider the exact sequence

$$0 \to R(-1)^4 \xrightarrow{M} R^4 \to \operatorname{Coker} M \to 0$$

Applying $\sim$ we obtain an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1)^4 \to \mathcal{O}_{\mathbb{P}^5}^4 \to \mathcal{F} \to 0 \qquad (12.5)$$

where $\mathcal{F} = \widetilde{\operatorname{Coker} M}$.

$\mathrm{Supp}(\mathcal{F}) = V(q)$

Consider the quadric hypersurface $X = V(q)$ and let $i : X \to \mathbb{P}^5$ denote the inclusion. Applying, i^* we arrive at an exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(-1)^4 \to \mathcal{O}_X^4 \to \mathscr{E} \to 0$$

where $\mathscr{E} = i^*\mathcal{F}$. (Recall that i^* is right-exact; the sequence here exact on the left because it is exact on stalks). Now the discussion above shows that $\mathscr{E}$ is locally free of rank 2 (as it has rank 2 at all closed points). The sheaf $\mathscr{E}$ is known as the *universal quotient bundle on the Grassmannian* $Gr(2,4)$.

12.9 *The Hilbert syzygy theorem*

Let k be a field and let $R = k[x_0, \ldots, x_n]$. Then if M is a finitely generated graded R-module, then *Hilbert Syzygy theorem* says that there is a finite free resolution (that is, an exact sequence)

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where $F^{b_k} = \bigoplus_{i=1}^{b_k} R(-d_i)$ is a free graded R-module. F_i is called the i-th *syzygy module* of the resolution. The minimal integer n that appears in such a resolution is called the *projective dimension of M*.

If we apply the $\sim$-functor here we obtain an exact sequence of sheaves on $\mathbb{P}^n_k$

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \widetilde{M} \to 0$$

where $\mathscr{E}_i = \bigoplus_{i=1}^{b_k} \mathcal{O}_{\mathbb{P}^n_k}(-d_i)$ is a direct sum of sheaves of the form $\mathcal{O}(d)$.

Thus any coherent sheaf can be resolved by locally free sheaves – in fact direct sums of invertible sheaves. This shows why the invertible sheaves $\mathcal{O}(d)$ are so important: They are the building blocks of all coherent sheaves on $\mathbb{P}^n$. We already saw some examples such a presentation was convenient. Let us give one more:

Example 12.24 *(The twisted cubic curve)* Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will consider the *twisted cubic curve* $C = V(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

i.e., $I = (q_0, q_1, q_2) = (x_1^2 - x_0x_2, x_0x_3 - x_1x_2, -x_2^2 + x_1x_3)$.

Consider the map of R-modules $R^3 \to I$ sending $e_i \mapsto q_i$. This is clearly surjective, since the q_i generate I. Let us consider the kernel of this map, that is, the module of relations of the form $a_0q_0 + a_1q_1 + a_2q_2 = 0$ for $a_i \in R$. There are two obvious relations of this form, i.e., the ones we get from expanding the determinants of the two matrices

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

(So first matrix gives $x_0q_2 - x_1q_1 + x_2q_2 = 0$ for instance). These give a map $R^2 \xrightarrow{\cdot M} R^3$, where M is the matrix above. This map is injective, and it turns out that there is an exact sequence of R-modules

$$0 \to R^2 \xrightarrow{M} R^3 \to I \to 0$$

Again, if we want to be completely precise, we should consider these as *graded* modules, so we must shift the degrees according to the degrees of the maps above

$$0 \to R(-3)^2 \xrightarrow{M} R(-2)^3 \to I \to 0$$

This gives the resolution of the ideal I of C. Then applying $\sim$, and using the fact that $\mathcal{I} = \widetilde{I}$, we get a resolution of the ideal sheaf of C:

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(-3)^2 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3_k}(-2)^3 \to \mathcal{I} \to 0$$

We will see later in Chapter 15 how to use sequences like this to extract geometric information about C. ☆