Chapter 13

First steps in sheaf cohomology

We have seen several examples of the global sections functor Γ failing to be exact when applied to a short exact sequence. On the other hand, we have a sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'')$$

which is exact at each stage except on the right.

Essentially, *cohomology* is a tool that allows us to continue this sequence, giving rise to a *long exact sequence* of cohomology:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathcal{F}') \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] & \Gamma(X,\mathcal{F}'') \arrow[dll, out=-10, in=170, looseness=1.5] \\
 & H^1(X,\mathcal{F}') \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{F}'') \arrow[dll, out=-10, in=170, looseness=1.5] \\
 & H^2(X,\mathcal{F}') \arrow[r] & H^2(X,\mathcal{F}) \arrow[r] & H^2(X,\mathcal{F}'') \arrow[r] & \cdots
\end{tikzcd}$$

In other words, the failure of surjectivity of the above is controlled by the group $H^1(X, \mathcal{F}')$ and the other groups in the sequence.

Cohomology groups can be defined in a completely general setting, for any topological space X and a (pre)sheaf $\mathcal{F}$ on it. With that as only input, we will define the *cohomology groups* $H^k(X,\mathcal{F})$, which will capture the main geometric invariants of $\mathcal{F}$. These should also be functorial in $\mathcal{F}$, which means that we want to construct functors

$$\begin{aligned} H^q(X,-)\colon \mathsf{AbSh}_X &\to \mathsf{Ab} \\ \mathcal{F} &\mapsto H^q(X,\mathcal{F}) \end{aligned}$$

$$\mathcal{F} \to \mathcal{G} \quad \rightsquigarrow \quad H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{G})$$

satisfying the following properties:

i) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$;

ii) A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces for all i group homomorphisms $H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$ which are functorial and takes the identity to the identity; in other words, each $H^i(X, -)$ is a functor;

iii) Every short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ gives a *long exact sequence* as above.

13.1 *Some homological algebra*

Complexes of groups

Recall that a *complex of abelian groups* $A^\bullet$ is a sequence of groups A^i together with maps between them

$$\begin{tikzcd} \cdots \arrow[r, "d^{i-2}"] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[rr, bend left, "0"] & A^i \arrow[r, "d^i"] & A^{i+1} \arrow[r, "d^{i+1}"] & \cdots \end{tikzcd}$$

such that $d^{i+1} \circ d^i = 0$ for each i. A *morphism of complexes* $A^\bullet \xrightarrow{f} B^\bullet$ is a collection such that $d^{i+1} \circ d^i = 0$ for each i. A *morphism of complexes* $A^\bullet \xrightarrow{f} B^\bullet$ is a collection of maps $f_p : A^p \to B^p$ making the following diagram commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f_{i-1}"] & A^i \arrow[r, "d^i"] \arrow[d, "f_i"] & A^{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

In this way, we can talk about kernels, images, cokernels, exact sequences of complexes, etc.

Z^p $\sigma \in A^p$ kosykel derom $\sigma \in \ker d^p$

B^p $\sigma \in A^p$ korand derom $\sigma = d^{p-1}\tau$ $\tau \in A^{p-1}$

We say that an element $\sigma \in A^p$ is a *cocycle* if it lies in the kernel of the map d^p i.e., $d^p\sigma = 0$. A *coboundary* is an element in the image of d^{p-1}, i.e., $\sigma = d^{p-1}\tau$ for some $\tau \in A^{p-1}$. These form subgroups of A^p, denoted by $Z^pA^\bullet$ and $B^pA^\bullet$ respectively. Since $d^p(d^{p-1}a) = 0$ for all a, all coboundaries are cocycles, so that $Z^pA^\bullet \supset B^pA^\bullet$. The *cohomology groups* of the complex $A^\bullet$ are set up to measure

$$\begin{tikzcd}
A^{p-1} \arrow[r, "d^{p-1}"] & A^p \arrow[r, "d^p"] & A^{p+1} \\
& \sigma \arrow[r] & 0 \\
\tau \arrow[r] & d\tau &
\end{tikzcd}$$

$Z^p A^\bullet \supset B^p A^\bullet$. The *cohomology groups* of the complex $A^\bullet$ are set up to measure the difference between these two notions. We the *p-th cohomology group* as the quotient group

$$H^p A^\bullet = Z^p(A^\bullet)/B^p(A^\bullet) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

måler hvor langt $A^\bullet$ er fra å være eksakt i grad p.

One thinks of $H^p A^\bullet$ as a group that measures the failure of the complex $A^\bullet$ of being exact at stage p: $A^\bullet$ is exact if and only if $H^p A^\bullet = 0$ for every p.

$$\begin{tikzcd}
0 \arrow[r] & F^{p-1} \arrow[r] \arrow[d] & G^{p-1} \arrow[r] \arrow[d] & H^{p-1} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & F^{p} \arrow[r] \arrow[d] & G^{p} \arrow[r] \arrow[d] & H^{p} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{p+1} \arrow[r] \arrow[d] & G^{p+1} \arrow[r] \arrow[d] & H^{p+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

The following fact is very important:

Proposition 13.1 *Suppose that* $0 \to F^\bullet \xrightarrow{f} G^\bullet \xrightarrow{g} H^\bullet \to 0$ *is an exact sequence of complexes. Then there is a* long exact sequence *of cohomology groups*

$$\begin{tikzcd}
\cdots \arrow[r] & H^p F^\bullet \arrow[r] & H^p G^\bullet \arrow[r] & H^p H^\bullet \arrow[dll, out=-10, in=170] & \\
 & H^{p+1} F^\bullet \arrow[r] & H^{p+1} G^\bullet \arrow[r] & H^{p+1} H^\bullet \arrow[r] & \cdots
\end{tikzcd}$$

$$
\begin{tikzcd}
Z^pF \arrow[d, phantom, "\parallel"] & Z^pG & Z^pH \\
0 \arrow[r] & \ker d^p \arrow[r] \arrow[d, hook] & \ker d^p \arrow[r] \arrow[d, hook] & \ker d^p \arrow[d, hook] \\
& {} & {} & {}
\end{tikzcd}
$$

Proof: For each $p \in \mathbb{Z}$, consider the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & F^p \arrow[r, "f_p"] \arrow[d, "d^p"'] & G^p \arrow[r, "g_p"] \arrow[d, "d^p"] & H^p \arrow[r] \arrow[d, "d^p"] & 0 \\
0 \arrow[r] & F^{p+1} \arrow[r, "f_{p+1}"] & G^{p+1} \arrow[r, "g_{p+1}"] & H^{p+1} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
F^p/F^{p+1} \arrow[r] & G^p/G^{p+1} \arrow[r] & H^p/H^{p+1}
\end{tikzcd}
$$

$$H^p = Z^p / B^p$$

where the rows are exact by assumption. By the Snake lemma, we obtain a sequence

$$\begin{tikzcd}
0 \arrow[r] & Z^p(F^\bullet) \arrow[r, "f_p"] & Z^p(G^\bullet) \arrow[r, "g_p"] & Z^p(H^\bullet) \arrow[dll, rounded corners, "\delta"'] & \\
& F^{p+1}/B^p(F^\bullet) \arrow[r, "f_{p+1}"] & G^{p+1}/B^p(G^\bullet) \arrow[r, "g_{p+1}"] & H^{p+1}/B^p(H^\bullet) \arrow[r] & 0
\end{tikzcd}$$

$H^p F^\bullet \qquad H^p G^\bullet \qquad H^p H^\bullet$

Consider now the diagram

$$\begin{tikzcd}
0 \arrow[r] & F^p/B^p(F^\bullet) \arrow[r, "f_p"] \arrow[d, "d^p"'] & G^p/B^p(G^\bullet) \arrow[r, "g_p"] \arrow[d, "d^p"] & H^p/B^p(H^\bullet) \arrow[r] \arrow[d, "d^p"] & 0 \\
0 \arrow[r] & Z^{p+1}(F^\bullet) \arrow[r, "f_{p+1}"] & Z^{p+1}(G^\bullet) \arrow[r, "g_{p+1}"] & Z^{p+1}(H^\bullet) &
\end{tikzcd}$$

where the rows are exact by the above. For the maps in this diagram, $H^pF^\bullet = \operatorname{Ker} d^p$ and $H^{p+1}F^\bullet = \operatorname{Coker} d^p$. Hence applying the Snake lemma one more time, we get the desired exact sequence. ❑

$H^{p+1} F^\bullet \longrightarrow H^{p+1} G^\bullet \longrightarrow H^{p+1} H$

Complexes of sheaves

The definitions and arguments of the previous subsection apply much more generally (to any abelian category). In particular, we make the following sheaf analogue. A *complex of sheaves* $\mathcal{F}^\bullet$ is a sequence of sheaves with maps between them

$$\cdots \xrightarrow{d^{i-2}} \mathcal{F}_{i-1} \xrightarrow{d^{i-1}} \mathcal{F}_i \xrightarrow{d^i} \mathcal{F}_{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^{i+1} \circ d^i = 0$ for each i. Given such a complex, we define the *cohomology sheaves* $H^p \mathcal{F}^\bullet$ as $\operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$. As above, a short exact sequence of complexes of sheaves gives rise to a long exact sequence of cohomology sheaves.

13.2 Čech cohomology

Let X be a topological space, and let $\mathcal{F}$ be a presheaf on it. Let $\mathcal{U} = \{U_i\}$ be an open cover of X, indexed by an ordered set I. As we saw previously, if $\mathcal{F}$ is a sheaf, the following sequence is exact:

$(t_i) \longrightarrow t_j - t_i$

$$0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j).$$

$s \longrightarrow s|_{U_i}$

$t_{ij} \mapsto t_{ij} - t_{ik} - t_{jk}$

$\longrightarrow \prod_{i,j,k} \mathcal{F}(U_i \cap U_j \cap U_k) \longrightarrow \dots$

The Čech complex is essentially the continuation of this sequence; it is a complex obtained by adjoining all the groups $\mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_r})$ over all intersections $U_{i_1} \cap \cdots \cap U_{i_r}$.

kompleks av grupper

Definition 13.2 *For a presheaf $\mathcal{F}$ on X, define the* Čech complex $C^\bullet(\mathcal{U},\mathcal{F})$ *of $\mathcal{F}$ (with respect to $\mathcal{U}$) as*

$$C^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{F}) \xrightarrow{d^2} \cdots$$

where

$$\begin{aligned} C^0(\mathcal{U},\mathcal{F}) &= \prod_{i_0\in I} \mathcal{F}(U_i) & (\sigma_i) \\ C^1(\mathcal{U},\mathcal{F}) &= \prod_{(i_0,i_1)\in I^2} \mathcal{F}(U_{i_0}\cap U_{i_1}) & (\sigma_{ij})_{ij} \\ &\vdots \\ C^p(\mathcal{U},\mathcal{F}) &= \prod_{(i_0,\dots,i_p)\in I^{p+1}} \mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_p}). & (\sigma_{i_0\cdots i_p}) \end{aligned}$$

$$\prod \mathcal{F}(U_i) \longrightarrow \prod \mathcal{F}(U_{ij}) \longrightarrow \prod \mathcal{F}(U_{ijk})$$

$$\sigma = \left(\sigma_{j_0 \cdots j_p} \right)_{j_0 \cdots j_p} \rightsquigarrow d\sigma \in C^{p+1}$$

and the coboundary map $d : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ *by*

$$(d^p\sigma)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots\hat{i}_j,\ldots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_p}}$$

where $i_0, \ldots \hat{i}_j, \ldots, i_{p+1}$ *means* $i_0, \ldots, i_{p+1}$ *with the index* i_j *omitted.*

EXAMPLE 13.3 Let us look at the first few maps in the complex:

- $d^0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$. If $\sigma = (\sigma_i)_i$, then

$$(d^0\sigma)_{ij} = \sigma_j - \sigma_i$$

$\sigma = (\sigma_i)$

$\sigma = (\sigma_{ij})_{i,j \in I}$

- $d^1 : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$. If $\sigma = (\sigma_{ij})_{\cancel{i}}$, then

$$(d^1\sigma)_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}$$

By direct substitution, we see that $d^1 \circ d^0 = 0$ (all the σ_{ij} cancel). This happens also in higher degrees by a basic computation using the definition of d^i.

Lemma 13.4 $d^{p+1} \circ d^p = 0$.

Utregning.

In particular, the $C^\bullet(\mathcal{U}, \mathcal{F})$ forms a *complex of abelian groups*. As before, we say that an element $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ is a *cocycle* if $d^p\sigma = 0$, and a *coboundary* if $\sigma = d^{p-1}\tau$, and denote these by Z^p and B^p respectively. The *Čech cohomology groups* of $\mathcal{F}$ are set up to measure the difference between these two notions:

Definition 13.5 *The p-th Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as*

$$H^p(\mathcal{U}, \mathcal{F}) = Z^p / B^p = (\operatorname{Ker} d^p) / (\operatorname{Im} d^{p-1})$$

It is not hard to check that a sheaf homomorphism $\mathcal{F} \to \mathcal{G}$ induces a mapping of Čech cohomology groups, so we obtain functors $\mathcal{F} \to H^p(\mathcal{U}, \mathcal{F})$ from abelian sheaves to abelian groups

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \longrightarrow \cdots$$

Example 13.6 Again it is instructive to consider the group $H^0(\mathcal{U}, \mathcal{F})$. Here the map $d^0 : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$, which is simply the usual map

$$\prod_i F(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

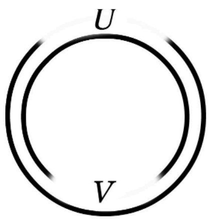
$$s_i \longrightarrow s_j - s_i$$

which has kernel $\mathcal{F}(X)$ by the sheaf axioms. It follows that $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$. ☆

$$H^0 = \frac{\ker d^0}{\operatorname{im} d^{-1}} = \ker d^0 = \mathcal{F}(X).$$

EXAMPLE 13.7 The most interesting cohomology group is arguably $H^1(\mathcal{U}, F)$. It is the group of cochains σ_{ij} such that $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$ modulo the cochains of the form $\sigma_{ij} = \tau_j - \tau_i$. . ☆

Example 13.8 Consider the unit circle $X = S^1$ (with the Euclidean topology), with its standard covering $\mathcal{U} = \{U, V\}$ (intersecting in two intervals) and let $\mathcal{F} = \mathbb{Z}_X$ (the constant sheaf).



$\prod \mathcal{F}(U_i)$ $\qquad\qquad$ $\prod \mathcal{F}(U_{ij})$

$\parallel$ $\qquad\qquad$ $\parallel$

Here we have

$$C^0(\mathcal{U},\mathcal{F}) = \mathbb{Z}_U \times \mathbb{Z}_V \qquad C^1(\mathcal{U},\mathbb{Z}) = \mathbb{Z}_{U\cap V} = \mathbb{Z}\times\mathbb{Z}$$

$0 \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$

$$(a,b) \longrightarrow (b-a,\, b-a)$$

The map $C^0 \to C^1$ is the map $d^0 : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $d^0(a,b) = (b-a, b-a)$. Hence

$$H^0(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Ker} d = \mathbb{Z}(1,1) \simeq \mathbb{Z}$$

and

$$H^1(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Coker} d = \frac{\mathbb{Z} \times \mathbb{Z}}{\mathbb{Z}(1,1)} \simeq \mathbb{Z}$$

H^0 : sammenhengskomponenter $\rightsquigarrow$ S^1 sml

H^1 : # "hull" $\rightsquigarrow$ S^1 har ett hull.

$U_0 \cap U_1 = \operatorname{Spec} k[t^{\pm 1}]$ $\qquad$ $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}$

EXAMPLE 13.9 Consider the projective line $\mathbb{P}^1_k$ covered by $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$. For the structure sheaf $\mathcal{F}$, the Čech -complex takes the following form:

C^0 $\qquad$ C^1

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(U_0) \times \mathcal{O}_{\mathbb{P}^1}(U_1) \arrow[r, "d"] & \mathcal{O}_{\mathbb{P}^1}(U_0 \cap U_1) \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & k[t] \times k[t^{-1}] \arrow[r] \arrow[u, "\simeq"'] & k[t^{\pm 1}] \arrow[r, "d"] \arrow[u, "\simeq"] & 0 \arrow[u]
\end{tikzcd}
$$

$r(t, t^{-1}) = a_n t^n + \dots + a_{-m} t^{-m}$

$(p(t), q(t^{-1})) \longrightarrow q(t^{-1}) - p(t)$

$H^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \ker d = k$

$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}) = \operatorname{coker} d = 0$

Where d sends a pair $(p(t), q(t^{-1}))$ to $q(t^{-1}) - p(t)$. As we saw in Chapter 5, $\operatorname{Ker} d = k$. It is on the other hand clear that any element of $k[t^{\pm 1}]$ can be written as a sum of a polynomial in t and one in t^{-1}, hence

$$H^1(\mathcal{U}, \mathcal{O}) = \operatorname{Coker} d = 0.$$

$$0 \longrightarrow k[t] \times k[t^{-1}] \xrightarrow{d} k[t^{\pm 1}] \longrightarrow 0$$

$$(p(t), \quad q(t^{-1})) \longmapsto t^n q(t^{-1}) - p(t)$$

$\tau_{10} : \mathcal{O}_{U_{01}} \xrightarrow{\cdot t^n} \mathcal{O}_{U_{01}}$

EXAMPLE 13.10 Continuing the above example, let us compute the sheaf cohomology groups for $\mathcal{F} = \mathcal{O}(n)$. The sequence takes the following form: $0 \longrightarrow k[t] \times k[t^{-1}] \xrightarrow{d} k[t^{\pm 1}] \longrightarrow 0$ where the map d is now given by

$$d(p(t), q(t^{-1})) = t^n q(t^{-1}) - p(t)$$

(see Section 5.3). As we computed in Proposition 5.2, we have $\operatorname{Ker} d \simeq k^{n+1}$ if $n \geqslant 0$, and $\operatorname{Ker} d = 0$ otherwise. The computation of H^1 is slightly more subtle.

$n \geq 0$

$$H^0(\mathcal{U}, \mathcal{O}(n)) = \ker d = \{ p(t) = t^n q(t^{-1}) \mid q \text{ polun} \}$$

$$= k\{1, t, \dots, t^n\}$$

$\rightsquigarrow \dim H^0(\mathcal{U}, \mathcal{O}(n)) = n+1$

$H^1(\mathcal{U}, \mathcal{O}(n)) = \operatorname{coker} d = 0 \qquad (n \geqslant 0)$

Consider first the case $n \geqslant 0$. As before, it is easy to see that any Laurent polynomial in $k[t, t^{-1}]$ can be written in the form $t^n q(t^{-1}) - p(t)$. In fact, this also works for $n = -1$, as $t^{-k} = t^{-1} \cdot t^{-k+1} - 0$ and $t^k = t^{-1} \cdot 0 - t^k$. Hence $H^1(\mathcal{U}, \mathcal{O}(n)) = 0$ for $n \geqslant -1$. For $n \leqslant -2$ however, any linear combination of the following monomials are not in the image:

$$t^{-1}, t^{-2}, \ldots, t^{n+1}$$

This implies that $H^1(\mathcal{U}, \mathcal{O}(n))$ is a k-vector space of dimension $-n+1$.

$n = -2$:

$$0 \longrightarrow k[t] \times k[t^{-1}] \longrightarrow k[t, t^{-1}] \longrightarrow 0$$

$$(p(t), q(t^{-1})) \xrightarrow{d} t^{-2} q(t^{-1}) - p(t)$$

$H^0 = \ker d = 0$

$H^1 = \operatorname{coker} d = k \cdot t^{-1} \simeq k.$

$H^1(\mathcal{U}, \mathcal{O}(-2)) \simeq k$

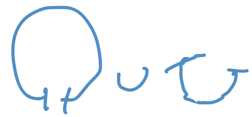
$n \gg 0 \to$ bare H^0

$n \ll 0 \to$ bare H^1

Proposition 13.11 *Let X be an* irreducible *topological space. Then for any covering $\mathcal{U}$ of X we have for a constant sheaf A_X*

$$H^p(\mathcal{U}, A_X) = 0$$

for $p > 0$.

Merk: S^1 ikke irredusibel $S^1 =$ 

ingen motsigelse for $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$

$X = \mathbb{P}^1$ $\qquad\qquad$ $p = [1,0]$

$Z =$ to punkter p, q $\qquad\qquad$ $q = [0,1]$

$Z = V(x_0 x_1)$ $\qquad\qquad$ $R = k[x_0, x_1]$

$$0 \to R(-2) \xrightarrow{\cdot x_0 x_1} R \to R/(x_0 x_1) \to 0$$

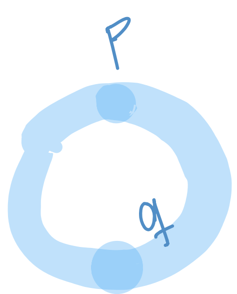
tilde:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Z \arrow[r] \arrow[d, equal] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] & i_* \mathcal{O}_Z \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-2) \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] & i_* \mathcal{O}_Z \arrow[r] & 0
\end{tikzcd}$$

$$0 \to \underset{\substack{\| \\ 0}}{H^0(\mathbb{P}^1, \mathcal{O}(-2))} \to \underset{\substack{\| \\ k}}{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})} \to H^0(Z, \mathcal{O}_Z)$$

$$\to \underset{\substack{\| \\ k t^{-1}}}{H^1(\mathbb{P}^1, \mathcal{O}(-2))} \to \underset{\substack{\| \\ 0}}{H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})} \to \cdots$$

$$0 \to \underset{\substack{\| \\ k}}{\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})} \xrightarrow{i^*} \overset{k^2}{\overset{\|}{\Gamma(Z, \mathcal{O}_Z)}} \to H^1(\mathbb{P}^1, \mathcal{O}(-2)) = k \to 0$$



$Z = p \cup q$

PROOF: In this case the Čech complex takes the form

$$\prod_{i\in I} A \to \prod_{i,j\in I} A \to \prod_{i,j,k\in I} A \to \cdots$$

Note that this complex of groups does not depend on X or the covering $\mathcal{U}$ – it is only the index set I which plays a role. In particular, the complex is the same as the Čech complex of A on a 1-point space (which makes it plausible that the higher cohomology should vanish). In this case it is easy to show by hand show that any p-cocycle is the boundary of some $(p-1)$-cochain for $p > 0$.

For instance, given a 1-cocycle $g = (g_{ij}) \in \prod_{i,j} A$, fix some $n \in I$ and define the element $h = (h_i) \in C^0(\mathcal{U}, A) = \prod_{i \in I} A$ by $h_i = g_{ni}$. The cocycle condition $0 = d^1(g)_{nij} = g_{ij} - g_{nj} + g_{ni}$ translates into $0 = g_{ij} - h_j + h_i$ or $g_{ij} = h_j - h_i$. This proves that the cocycle $g = (g_{i,j})$ is the coboundary of the element $h = (h_i)$, and thus that the class of that cocycle is zero in $H^1(\mathcal{U}, A_X)$.

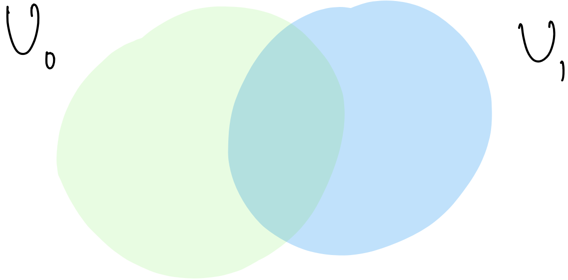
Sheaf cohomology

As seen in the examples above, the groups $H^p(\mathcal{U}, \mathcal{F})$ are easily computable, if one is given a nice cover of X. Indeed, the maps in the Čech complex are completely explicit, and computing their kernels and images involve only basic row operations from linear algebra.

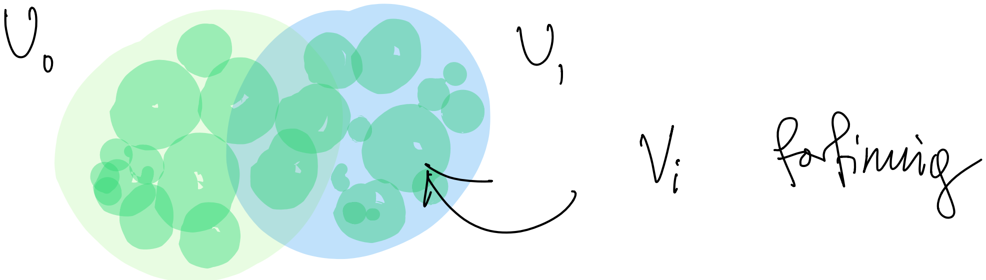
problem: kan velge $\mathcal{U} = \{X\}$

$\rightsquigarrow$ Čech kompleks $\quad 0 \to \mathcal{F}(X) \to 0$

On the other hand, the definition of the cohomology groups is a little bit unsatisfactory for various reasons. First of all, the groups $H^p(\mathcal{U}, \mathcal{F})$ depend on the cover $\mathcal{U}$, whereas we want something canonical that only depends on $\mathcal{F}$. More importantly, it is not clear that the definition above really captures enough of the desired information about $\mathcal{F}$. For instance, $\mathcal{U}$ could consist of the single open set X, and so $H^i(\mathcal{U}, \mathcal{F}) = 0$ for all $i \geqslant 1$! Finally, it is not at all clear if these groups satisfy the requirements mentioned in the introduction.



There is a fix for all of these problems which involves passing to finer and finer ‘refinements’ of the covering. We say that a covering $\mathcal{V} = \{V_j\}_{j\in J}$ is a *refinement* of $\mathcal{U} = \{U_i\}_{i\in I}$ if for every $V_j \in \mathcal{V}$, there is a $i \in I$ so that $V_j \subset U_i$. This defines a partial ordering on the coverings which we denote by $\mathcal{V} \leqslant \mathcal{U}$. If we



fix a map $\epsilon : J \to I$ so that $V_j \subset U_{\sigma(j)}$ for every j, we can define a *refinement homomorphism*

$$\mathrm{ref}_{\mathcal{U},\mathcal{V}} : H^p(\mathcal{U},\mathcal{F}) \to H^p(\mathcal{V},\mathcal{F})$$

by setting

$$(\mathrm{ref}_{\mathcal{U},\mathcal{V}}(\sigma))_{j_0,\ldots,j_p} = \left(\sigma_{\epsilon j_0,\ldots,\epsilon j_p}\right)|_{V_{j_0 \cap \cdots \cap j_p}}$$

One computes easily that $d \circ \mathrm{ref} = \mathrm{ref} \circ d$, so that ref induces a map on cohomology groups. Moreover, one can check that while the refinement depends on the choice of the function $\epsilon : J \to I$, the map ref on cohomology does not.

One can then define a group $H^p(X, \mathcal{F})$ to be the direct limit of all $H^p(\mathcal{U}, \mathcal{F})$ as $\mathcal{U}$ runs through all possible open covers $\mathcal{U}$ ordered by $\leqslant$. The resulting groups are indeed canonical, and turn out to give the right answer for cohomology:

DEFINITION 13.12 *The groups $H^p(X, \mathcal{F})$ are called the* cohomology groups of $\mathcal{F}$. *In symbols,*

$$H^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F})$$

avhenger ikke av overdekningen

$\therefore$ helt kanonisk

The main preoperties of Čech cohomology are summarized in the following theorem:

THEOREM 13.13 *Let X be a topological space and let $\mathcal{F}$ be a sheaf on X.*

- *The Čech cohomology groups are functors $H^i(X,-) : \mathsf{Sh}_X \to \mathsf{Groups}$.*
- $H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) = \mathcal{F}(X)$,
- *Short exact sequences of sheaves induce long exact sequences of cohomology.*
- *(Leray's theorem). If $\mathcal{F}$ is a sheaf and $\mathcal{U}$ is a covering such that $H^i(U_{i_1} \cap \cdots U_{i_p}, \mathcal{F}) = 0$ for all $i > 0$ and multiindexes $i_1 < \cdots < i_p$, then*

$$H^i(X,\mathcal{F}) = H^i(\mathcal{U},\mathcal{F}).$$

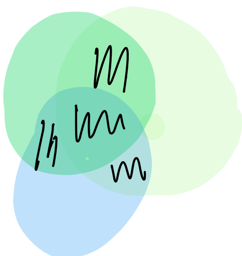
∴ Dersom $\mathcal{U}$ er en "pen" overdeling $\Rightarrow$ $H^i(\mathcal{U},\mathcal{F})$ beregner $H^i(X,\mathcal{F})$

Dersom X er separert

$\Longrightarrow$ alle affine overdekninger $\mathcal{U} = \{U_i\}$

viktig! $\longrightarrow$ tilfredsstiller denne betingelsen.

The last statement (Leray's theorem) is very important. It says that even though $H^i(X, \mathcal{F})$ is defined as an infinite directed limit over coverings $\mathcal{U}$, it suffices to compute it at a covering which is 'sufficiently fine' in the sense that the higher groups $H^i(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0$ vanish for $i > 0$. In practice, the latter condition is rather easy to check: It holds for instance if all of the intersections are affine schemes (see Corollary 14.2).



$p = 1$

$U_0 \cap U_1$

$U_0 \cap U_2$

$U_1 \cap U_2$

Skal vise: $\boxed{X = \operatorname{Spec} A \quad \mathcal{F} = \tilde{M} \rightsquigarrow H^i(X, \mathcal{F}) = 0 \quad i > 0.}$

X separert: Snittet av affine $U_i \subset X$ er affint $\to$ ok.

The long exact sequence for quasi-coherent sheaves

Let X be a scheme and consider a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

If $U = \operatorname{Spec} A$ is affine, proved in Proposition ??? that the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0. \tag{13.1}$$

is exact. This means that if we have an affine cover $\mathcal{U} = \{U_i\}_{i \in I}$ with the property that each intersection

$$U_{i_1} \cap \cdots \cap U_{i_p}$$

$\prod \mathcal{F}(U_{i_0 \cdots i_{p+1}})$

is affine, then we have an exact sequence

$$0 \longrightarrow C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{G}) \longrightarrow C^p(\mathcal{U}, \mathcal{H}) \longrightarrow 0.$$

and consequently, the sequence of *Čech complexes*

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{G}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{H}) \longrightarrow 0.$$

is also exact. Thus we are in position to apply Lemma 13.1 to get a long exact sequence of Čech cohomology groups

$$\cdots \longrightarrow H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}, \mathcal{G}) \longrightarrow H^i(\mathcal{U}, \mathcal{H}) \longrightarrow \cdots .$$

If X is separated, such coverings are cofinal in the directed system of coverings, so we in fact get a proof of the long exact sequence (13.1) for quasi-coherent sheaves.

In general, it can certainly happen that the restriction map (13.1) is *not* surjective – one can for instance take the open covering of X with just one open set X. This explains why the Čech cohomology groups $H^i(\mathcal{U}, \mathcal{F})$ do not give long exact sequences in general. However, by passing to smaller refinements $\mathcal{V} \leqslant \mathcal{U}$, we can arrange that any section lifts and we can use the above to construct δ.