To hovedteoremer:

1) $H^i(\operatorname{Spec} A, \mathcal{F}) = 0$ $\forall i > 0$
$\mathcal{F}$ kvasikoherent

2) $H^i(\mathbb{P}^n_A, \mathcal{O}(d)) = ?$ n, d

Chapter 14

Computations with cohomology

Čech kohomologi

$$H^i(X, \mathcal{F}) := \varinjlim_{\mathcal{V}} H^i(\mathcal{V}, \mathcal{F})$$

$\mathcal{V}$ åpne overdekninger av X

1) + Leray $\Rightarrow$ $H^i(X, \mathcal{F}) = H^i(\mathcal{V}, \mathcal{F})$
X separert for en affin overdekning $\mathcal{V}$.

14.1 *Cohomology of sheaves on affine schemes*

THEOREM 14.1 *Let $X = \operatorname{Spec} A$ and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then*

$$H^p(X, \mathcal{F}) = 0 \text{ for all } p > 0.$$

$\|$

$$\varinjlim H^i(\mathcal{U}, \mathcal{F})$$

nothing !

PROOF: Recall that we defined the groups $H^i(X, \mathcal{F})$ by taking the direct limit of $H^i(\mathcal{U}, \mathcal{F})$ over finer and finer coverings $\mathcal{U}$ of X. Since the distinguished opens subsets form a basis for the topology on X. It suffices to prove that

$$\boxed{H^p(\mathcal{U}, \mathcal{F}) = 0 \text{ for all } p > 0.}$$

for a covering $\mathcal{U} = \{D(g_i)\}$ where the g_i are finitely many elements of A generating the ring. (We can choose finitely many g_i, by the quasi-compactness of X).

$U_i = D(g_i)$

$$0 \to \mathcal{F}(X) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_{ij}) \to \prod \mathcal{F}(U_{ijk}) \to \cdots$$

$$\begin{array}{ccccccc} \| & & \| & & \| & & \| \\ M & \longrightarrow & M_{g_i} & \longrightarrow & M_{g_i g_j} & \longrightarrow & M_{g_i g_j g_k} \end{array}$$

As $\mathcal{F}$ is quasi-coherent, we may write $\mathcal{F} = \widetilde{M}$ for some A-module M, and $M = \Gamma(X, \mathcal{F})$. With this setup, the fact that Čech cohomology vanishes of a quasi-coherent module on an affine scheme corresponds to the observation that for a commutative ring A; a finite sequence of elements $(g_i)_{i \in I}$ of A generating A as an ideal; and some A-module M, the following sequence is exact:

$$0 \to M \to \prod_{i \in I} M_{g_i} \to \prod_{i,j \in I} M_{g_i g_j} \to \prod_{i,j,k \in I} M_{g_i g_j g_k} \to \cdots$$

eksakt her ?

$$0 \to M \to \prod_{i \in I} M_{g_i} \to \prod_{i,j \in I} M_{g_i g_j} \to \prod_{i,j,k \in I} M_{g_i g_j g_k} \to \dots$$

Here the boundary maps are given as alternating sums of localization maps. For example,

$$d : \prod_{i,j \in I} M_{g_i g_j} \to \prod_{i,j,k \in I} M_{g_i g_j g_k}$$

maps $(\sigma_{ij})_{i,j} \in M_{g_i g_j}$ to $(\sigma_{jk} - \sigma_{ik} + \sigma_{ij})_{i,j,k}$.

$$\left(\frac{m_{ij}}{(g_i g_j)^N} \right) \longrightarrow \frac{g_i^N m_{jk} - g_j^N m_{ik} + m_{ij} g_k^N}{(g_i g_j g_k)^N}$$

$M_g \to M_h$

Notice that the beginning of the exact sequence

$$0 \to M \to \prod_i M_{g_i} \to \prod_{i,j} M_{g_i g_j}$$

appeared in the construction of the quasi-coherent module $\widetilde{M}$. The proof for the exactness of this sequence is similar to the general case.

To prove that the cohomology groups vanish, we must to each cocycle σ (such that $d\sigma = 0$) find an element τ such making $\sigma = d\tau$ a boundary. The proof is a direct calculation; one constructs an element τ by hand.

$p=1$ $\quad H^1(\mathcal{U}, \mathcal{F}) = 0$

$$0 \to M \to \prod_{i \in I} M_{g_i} \to \prod_{i,j \in I} M_{g_i g_j} \to \prod_{i,j,k \in I} M_{g_i g_j g_k} \to \dots$$

$\exists ? \tau \qquad \sigma = (\sigma_{ij}) \longrightarrow 0$

To see how this can be done, let us for simplicity consider the case $p = 1$ first. Let $\sigma \in \prod_{i,j} M_{g_i g_j}$ be in the kernel of d. We may write

$$\sigma_{ij} = \frac{m_{ij}}{(g_i g_j)^r} \text{ where } m_{ij} \in M$$

for some r (since the index set I is finite, we may choose this independent of i, j).

The relation $d\sigma = 0$ gives the relation

$$(d\sigma)_{ijk} = \frac{m_{jk}}{(g_jg_k)^r} - \frac{m_{ik}}{(g_ig_k)^r} + \frac{m_{ij}}{(g_ig_j)^r} = 0 \qquad \forall i,j,k$$

in $M_{g_ig_jg_k}$. In other words, we have, in $M_{g_jg_k}$,

$$\frac{g_i^{r+l} m_{jk}}{(g_jg_k)^r} = \frac{g_i^l g_j^r m_{ik}}{(g_jg_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_jg_k)^r} \tag{14.1}$$

$$1 = \sum h_i g_i^{r+l}$$

for some $l \geqslant 0$. Now, as the open sets $D(g_i) = D(g_i^{r+l})$ cover X, we have a relation

$$1 = \sum_{i \in I} h_i g_i^{r+l}$$

where $h_i \in A$. Let us define $\tau = (\tau_j) \in \prod M_{g_j}$ by

$$\tau_j = \sum_{i \in I} h_i \frac{g_i^l m_{ij}}{g_j^r}$$

In $\prod M_{g_j g_k}$, we may write this as

$$\tau_j = \sum_{i \in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r}$$

We want to show that $d\tau = \sigma$. This is a basic computation, using the relation

(14.1) above:

$$\begin{aligned}
(d\tau)_{jk} &= \tau_k - \tau_j \\
&= \sum_{i\in I} h_i \frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \sum_{i\in I} h_i \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \\
&= \sum_{i\in I} h_i \left(\frac{g_i^l g_j^r m_{ik}}{(g_j g_k)^r} - \frac{g_i^l g_k^r m_{ij}}{(g_j g_k)^r} \right) \\
&= \sum_{i\in I} h_i \frac{g_i^{r+l} m_{jk}}{(g_j g_k)^r} = \frac{m_{jk}}{(g_j g_k)^r} \sum_{i\in I} h_i g_i^{r+l} \\
&= \frac{m_{jk}}{(g_j g_k)^r} = \sigma_{jk}
\end{aligned}$$

As desired. Hence $H^1(\mathcal{U}, \mathcal{F}) = 0$.

Čech cohomology and affine coverings

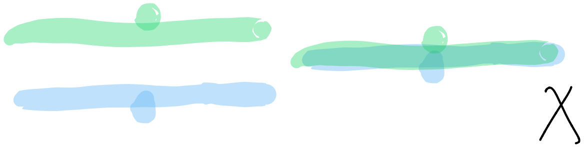
As a Corollary of the previous theorem, we see that affine coverings of schemes satisfy the conditions of Leray's theorem (see Theorem 13.13). This implies

Corollary 14.2 *Let X be a noetherian scheme and let $\mathcal{U} = \{U_i\}$ be an affine covering such that all intersections $U_{i_0} \cap \cdots \cap U_{i_s}$ are affine. Then*

$$H^i(X, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F})$$

In particular, the theorem applies to *any* affine covering on a noetherian separated scheme.

$H^0(X, \mathcal{O}) = A[u]$



$H^1(X, \mathcal{O}) = A[u^{\pm 1}]/A[u] = Au^{-1} \oplus Au^{-2} \oplus \cdots$

Example 14.3 Consider the 'affine line with two origins' X from Example 5.4. X is covered by two affine subsets $X_1 = \operatorname{Spec} A[u]$ and $X_2 = \operatorname{Spec} A[u]$ and these are glued along their common open set $X_{12} = D(u) = \operatorname{Spec} A[u, u^{-1}]$. The Čech complex looks like

$p(u) \quad q(u) \longrightarrow q(u) - p(u)$

$$0 \longrightarrow A[u] \times A[u] \xrightarrow{d^1} A[u, u^{-1}] \xrightarrow{d^2} 0$$

As we checked in the example, $\mathcal{O}_X(X) = \operatorname{Ker} d^1 = A[u]$. More strikingly, the cokernel $H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^1$ of the map $A[u] \oplus A[u] \to A[u, u^{-1}]$ is $A[u, u^{-1}]/A[u]$, so $H^1(X, \mathcal{O}_X)$ is not finitely generated as an A-module. This gives another proof that X is not isomorphic to an affine scheme. ☆

14.2 Cohomology and dimension

Theorem 14.4 *Let X be a topological space of dimension n, and let $\mathcal{F}$ be a sheaf on X. Then*

$$H^p(X, \mathcal{F}) = 0$$

for all $p > n$.

ex $X = \mathbb{P}^1$ H^0, H^1

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow 0$$

We will prove this in a special case, namely for a quasi-projective scheme.

Lemma 14.5 *Let X be a topological space and let $Z \subset X$ be a closed subset. Then for any abelian sheaf $\mathcal{F}$ on Z, we have $H^p(Z, \mathcal{F}) = H^p(X, i_*\mathcal{F})$.*

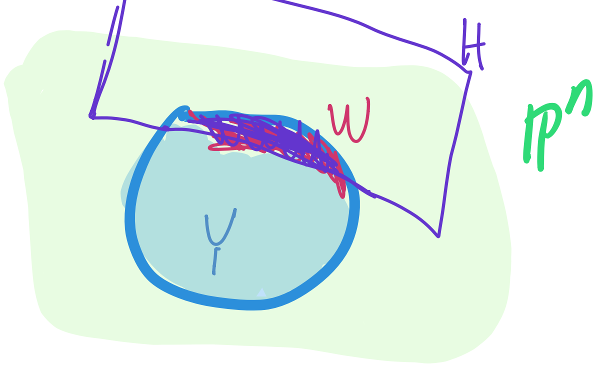
Proof: This follows from the basic fact that for $U \subset X$ open $\Gamma(U, i_*\mathcal{F}) = \Gamma(Z \cap U, \mathcal{F})$, so the two cohomolgy groups arise from the same Čech complexes ∎

Theorem 14.6 *Let X be a quasi-projective scheme of dimension n. Then X admits an open cover $\mathcal{U}$ consisting of at most $n+1$ affine open subsets.*

This implies that

$$H^i(X, \mathcal{F}) = 0 \text{ for } i > n$$

for any quasi-coherent sheaf $\mathcal{F}$ on X.



PROOF: Let X be a quasi-projective scheme, i.e., $X = Y - W$, where $Y, W \subset \mathbb{P}^n_A$ are closed subschemes. Consider the irreducible decomposition $Y = \bigcup_i Y_i$ and observe that $I_W \nsubseteq \bigcup I_{Y_i}$ where $I_{\cancel{T}W} \subset A[x_0, \ldots, x_N]$ denotes the ideal of the set $\cancel{T}W \subset \mathbb{P}^N$. Pick a homogenous polynomial f of degree d such that $f \in I_W - (\cup_i I_{Y_i})$. Let $H = Z(f)$. Then $\mathbb{P}^n - H = D_+(f)$ is affine and hence so is $Y - H$ (being a closed subscheme of an affine scheme).

By construction $Y - H \subset Y - W = X$ and $H \not\supseteq Y_i$ for any i by the choice of f. Therefore $\dim(Y_i \cap H) < \dim Y_i$ so we may use induction on $\dim X$. Notice that the affine subset is obtained as an affine subset of the ambient projective space intersected with our scheme, so the affine schemes obtained subsequently are restrictions of affine subschemes of the original X. This shows the first claim.

$$0 \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

For the second, note that for $\leqslant n+1$ affines, there are at most n $+1$ terms $C^i(X, \mathcal{F})$ in the Čech complex. From this it follows that $0 = H^i(\mathcal{U}, \mathcal{F}) = H^i(X, \mathcal{F})$ for any $\mathcal{F}$ and $i > n$. ❑

14.3 *Cohomology of sheaves on projective space*

On projective space $\mathbb{P}^n_A$ we have a distinguished covering via the open sets $D_+(x_i)$. We can use the Čech complex associated to this covering to compute the cohomology of the invertible sheaves $\mathcal{O}(m)$ on $\mathbb{P}^n_A$.

Theorem 14.7 *Let $X = \mathbb{P}^n_A = \operatorname{Proj} R$ where $R = A[x_0, \ldots, x_n]$ where A is a noetherian ring.*

i)

$$\Gamma(X, \mathcal{O}(m)) = H^0(X, \mathcal{O}_X(m)) = \begin{cases} R_m \text{ for } m \geqslant 0 \\ 0 \text{ otherwise} \end{cases}$$

homogene polynomer av grad m

ii)

$$H^n(X, \mathcal{O}_X(m)) = \begin{cases} A \text{ for } m = -n-1 \\ 0 \text{ for } m > -n-1 \end{cases}$$

$H^1(\mathbb{P}^1, \mathcal{O}(-2)) = A$

iii) For $m \geqslant 0$, there is a perfect pairing[1] of A-modules

$$H^0(X, \mathcal{O}_X(\underbrace{m}_{= R_m})) \times H^n(X, \mathcal{O}_X(-m-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) \simeq A$$

iv) For $0 < i < n$ and all $m \in \mathbb{Z}$, we have

$$H^i(X, \mathcal{O}_X(m)) = 0$$

$\dim H^n(\mathcal{O}(-m-n-1))$

$= \dim R_m$

$= \binom{m+n-1}{n-1}$

PROOF: Consider the $\mathcal{O}_X$-module $\mathcal{F} = \bigoplus_{m\in\mathbb{Z}} \mathcal{O}(m)$. We would like to show for instance that $H^i(X, \mathcal{F}) = 0$ for $i \neq 0, n$; this is equivalent to $H^i(X, \mathcal{O}_X(m)) = 0$ for all m, but $\mathcal{F}$ has the advantage that it is a graded $\mathcal{O}_X$-*algebra*. Consider

$$\Gamma(U_i, \mathcal{F}) = \Gamma\left(U_i, \bigoplus \mathcal{O}(m)\right)$$

$$= \bigoplus \Gamma(U_i, \mathcal{O}(m))$$

$$= \bigoplus_{m \in \mathbb{Z}} \left((R(m))_{x_i} \right)_0 = R_{x_i}$$

the Čech complex of $\mathcal{F}$ associated with the standard covering $\mathcal{U} = \{U_i\}$ where $U_i = D_+(x_i) = \operatorname{Spec} R_{(x_i)}$. This is simply

$$\underbrace{\prod_i R_{x_i}}_{C^0} \xrightarrow{d^0} \underbrace{\prod_{i,j} R_{x_i x_j}}_{C^1} \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \underbrace{R_{x_0 \cdots x_n}}_{C^n}$$

We have a graded isomorphism of R-modules:

$$\begin{aligned} H^0(X, \mathcal{F}) &= \operatorname{Ker} d^0 \\ &= \{(r_i)_{i \in I} | r_i \in R_{x_i}, r_i = r_j \in R_{x_i x_j}\} \\ &\simeq R. \end{aligned}$$

This isomorphism preserves the grading, so we get (i).

$$0 \to C^0 \to \cdots \to C^{n-1} \to C^n \to 0$$

$$C^{n-1} = \prod_i R_{x_0 \cdots \hat{x_i} \cdots x_n}, \qquad C^n = R_{x_0 \cdots x_n}$$

For (ii): Note that $R_{x_0 \cdots x_n}$ is a free graded A-module spanned by monomials of the form

$$x_0^{a_0} \cdots x_n^{a_n}$$

for multidegrees $(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$. The image of d^{n-1} is spanned by such monomials where at least one a_i is non-negative. Hence

$$\begin{aligned} H^n(X, \mathcal{F}) &= \operatorname{Coker} d^{n-1} \\ &= A\left\{x_0^{a_0} \cdots x_n^{a_n} | a_i < 0 \forall i\right\} \subset R_{x_0 \ldots x_n} \end{aligned}$$

$n=1$:

$$0 \to R_{x_0} \oplus R_{x_1} \longrightarrow R_{x_0 x_1} \longrightarrow 0$$

$$(x_0 x_1)^{-1}$$

Hence

$$\begin{aligned} H^n(X, \mathcal{O}_X(m)) &= H^n(X, \mathcal{F})_m \\ &= A\left\{x_0^{a_0} \cdots x_n^{a_n} | a_i < 0 \forall i, \sum a_i = m\right\} \subset R_{x_0 \ldots x_n} \end{aligned}$$

In degree $-n-1$ there is only one such monomial, namely $x_0^{-1} \cdots x_n^{-1}$.

(iii): If we identify $H^n(X, \mathcal{O}_X(-m-n-1))$ with

$$A\left\{x_0^{a_0}\cdots x_n^{a_n} : a_i < 0\forall i, \sum a_i = m\right\}$$

and $H^0(X, \mathcal{O}(m)) = R_m$, we can define the pairing via multiplication of Laurent polynomials:

$$\begin{aligned} H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) &\to R_{x_0\cdots x_n} \\ (x_0^{m_0}\cdots x_n^{m_n}) \times (x_0^{a_0}\cdots x_n^{a_n}) &\mapsto x_0^{a_0+m_0}\cdots x_n^{a_n+m_n} \end{aligned}$$

Here the exponents satisfy $m_i \geqslant 0$, $a_i < 0 \sum a_i = -m-n-1$, $\sum m_i = m$. This gives a map

$$H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) = Ax_0^{-1} \cdots x_n^{-1}$$

sending $(x_0^{m_0} \cdots x_n^{m_n}) \times (x_0^{a_0} \cdots x_n^{a_n})$ to zero if $m_i + a_i \geqslant 0$ for some i. This pairing is perfect: The dual of a monomial $(x_0^{m_0} \cdots x_n^{m_n})$ is represented by $(x_0^{-m_0-1} \cdots x_n^{-m_n-1})$.

$H^i(\mathbb{P}^n_A, \mathcal{O}(m)) = 0 \quad \forall m \in \mathbb{Z}$

$1 \leq i \leq n-1$

(iv) This point is more involved, and we proceed by induction on n. For $n = 1$, there is nothing to prove. For $n > 1$, let $H = V(x_n) \simeq \mathbb{P}^{n-1}$ be the hyperplane determined by x_n. We have an exact sequence

$$0 \to R(-1) \xrightarrow{\cdot x_n} R \to R/(x_n) \to 0 \tag{14.3}$$

I_H $\quad$ $H \simeq \mathbb{P}^{n-1}_{\Lambda}$

$\shortparallel$

Applying $\sim$, we find

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_*\mathcal{O}_H \to 0 \qquad \otimes\, \mathcal{O}(m)$$

where $i : H \to X$ is the inclusion. If we take the direct sum of all the twists of this sequence, we get

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to i_*\mathcal{F}_H \to 0$$

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}(m) \longrightarrow i_* \mathcal{O}_H \otimes \mathcal{O}(m) \longrightarrow 0$$

$$i_* \mathcal{O}_H \otimes \mathcal{O}(m) = i_*(\mathcal{O}_H(m))$$

$$H^i(H, \mathcal{F}_H) = 0 \qquad 1 \leq i \leq n-2$$

where $\mathcal{F}_H = \bigoplus_{m\in\mathbb{Z}} \mathcal{O}_H(m)$. By induction on n, we have for $0 < i < n-1$ and all $m \in \mathbb{Z}$: $H^i(X, i_*\mathcal{O}_H(m)) = H^i(H, \mathcal{O}_H(m)) = 0$. So taking the long exact seqence of cohomology, we get isomorphisms

$$H^i(X, \mathcal{F}(-1)) \xrightarrow{\cdot x_n} H^i(X, \mathcal{F})$$

for $1 < i < n-1$. We claim that we have isomorphisms also for $i = 1$ and $i = n-1$. For $i = 1$, this follows because the sequence

$$0 \to H^0(X, \mathcal{F}(-1)) \to H^0(X, \mathcal{F}) \to H^0(X, i_*\mathcal{F}_H) \to 0$$

is exact (this is the same sequence as (14.3)).

For $i = n-1$ we need to show that

$$0 \to H^{n-1}(X, i_*\mathcal{F}_H) \xrightarrow{\delta} H^n(X, \mathcal{F}(-1)) \xrightarrow{\cdot x_n} H^n(X, \mathcal{F})$$

is exact. The kernel of $\cdot x_n$, is generated by monomials $x_0^{a_0} \cdots x_n^{a_n}$ with $a_i < 0$ for all i. So it suffices to show that the connecting map δ is just multiplication by x_n^{-1}. Define $R' = R/x_n$. Writing the arrows in the Čech complex, vertically we get the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \prod_i R(-1)_{x_0 \cdots \hat{x}_i \cdots x_n} \arrow[r, "\cdot x_n"] \arrow[d] & \prod_i R_{x_0 \cdots \hat{x}_i \cdots x_n} \arrow[r] \arrow[d] & R'_{x_0 \cdots x_{n-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R_{x_0 \cdots x_n}(-1) \arrow[r, "\cdot x_n"] & R_{x_0 \cdots x_n} \arrow[r] & 0 &
\end{tikzcd}$$

Skal vise $H^i(\mathbb{P}^n_A, \mathcal{F}) = 0 \quad 0 < i < n$,

$$H^i(\mathbb{P}^n_A, \mathcal{F}) \xrightarrow{\cdot x_n} H^i(\mathbb{P}^n_A, \mathcal{F}) \quad \text{iso}$$

vil vise at denne er 0 som A-modul

If $x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}$ is a monomial in $H^{n-1}(H, \mathcal{F}_H)$ with $a_i < 0$ for all $1 \leqslant i \leqslant n-1$, then it comes from an $(n+1)$-tuple in $\prod_i R_{x_0 \cdots \hat{x}_i \cdots x_n}$ which maps to $\pm x_0^{a_0} \cdots x_{n-1}^{a_{n-1}}$ in $R_{x_0 \cdots x_n}$, which is in turn mapped onto by the monomial $x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} x_n^{-1}$ in $R_{x_0 \cdots x_n}(-1)$. So $\delta(x_0^{a_0} \cdots x_{n-1}^{a_{n-1}})$ is represented by the monomial $x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} x_n^{-1}$ in $H^n(X, \mathcal{F}(-1))$.

$$U_n = \mathbb{P}^n - V(x_n) \simeq \mathbb{A}^n$$

Now we claim that we have an isomorphism $H^*(X, \mathcal{F})_{x_n} = H^*(U_n, \mathcal{F}|_{U_n})$. Indeed, the Čech complex of $\mathcal{F}|_{U_n}$ with respect to the covering $U_i \cap U_n$ is just the

$$U_n = \operatorname{Spec}(R_{x_n})_0$$

$$\Gamma(U_n, \mathcal{F}) = \mathcal{F}(X)_{x_n} \qquad \mathcal{F} = \widetilde{\mathcal{F}(X)}$$

localization of $C^\bullet(X, \mathcal{F})$ at x_n. Localization is exact, so it preserves cohomology, which gives the claim.

We know that $H^i(X, \mathcal{F})_{x_n} = H^i(U_n, \mathcal{F}|_{U_r}) = 0$ for all $i > 0$, since U_n is affine. Hence for $l \gg 0$, $x_n^l H^i(X, \mathcal{F}) = 0$ as an A-module. However, we have shown that $\cdot x_n$ gives an isomorphism of $H^i(X, \mathcal{F})$ for $0 < i < n$. This implies that $H^i(X, \mathcal{F}) = 0$. ❑

COROLLARY 14.8 *Let k be a field. Then for* $m \geqslant 0$

- $\dim_k H^0(\mathbb{P}^n_k, \mathcal{O}(m)) = \binom{m+n}{n}$ $(\neq 0 \quad m \gg 0)$
- $\dim_k H^n(\mathbb{P}^n_k, \mathcal{O}(-m)) = \binom{m-1}{n}$ $(\neq 0 \quad m \ll 0)$

and all other cohomology groups are 0.

Recap $H^i(\mathbb{P}^n_A, \mathcal{O}(d)) = 0$ dersom $0 < i < n$

$= 0$ dersom $i = 0$ og $d < 0$

$h^0(\mathbb{P}^n, \mathcal{O}(d)) = \binom{n+d}{d}_{d \geq 0}$ $= 0$ dersom $i = n$ og $d \geq -n-1$

$H^n(\mathbb{P}^n, \mathcal{O}(-d-n-1)) \cong H^0(\mathbb{P}^n, \mathcal{O}(d))^\vee$

14.4 Extended example: Plane curves

Let $X = V(f) \subset \mathbb{P}^2_k$ be a plane curve, defined by an homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d. Let us compute the groups of the structure sheaf $H^i(X, \mathcal{O}_X)$. We have the ideal sheaf sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X \to 0$$

Vil regne ut: $H^0(X, \mathcal{O}_X)$ og $H^1(X, \mathcal{O}_X)$

Defn Genuset til X defineres som $g = h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X)$

Tidligere lemma $\Rightarrow$ $H^i(X, \mathcal{O}_X) = H^i(\mathbb{P}^2, \underbrace{i_* \mathcal{O}_X}_{\text{koherent knippe på } \mathbb{P}^2_k.})$

where the ideal sheaf $\mathcal{I}_X$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X$. By Section 12.8, $\mathcal{O}_{\mathbb{P}^2}(-X) \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$, and the sequence can be rewritten as

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0.$$

tilde

$$0 \to R(-d) \to R \to R/f \to 0$$

From the short exact sequence, we get the long exact sequence as follows:

$$
\begin{aligned}
0 \longrightarrow \overset{=0}{H^0(\mathbb{P}^2, \mathcal{O}(-d))} \longrightarrow \overset{= k}{H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})} \longrightarrow H^0(X, \mathcal{O}_X) \\
\longrightarrow \overset{=0}{H^1(\mathbb{P}^2, \mathcal{O}(-d))} \longrightarrow \overset{=0}{H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})} \longrightarrow H^1(X, \mathcal{O}_X) \\
\longrightarrow H^2(\mathbb{P}^2, \mathcal{O}(-d)) \longrightarrow \underset{=0}{H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})} \longrightarrow 0.
\end{aligned}
$$

$$0 \to k \xrightarrow{\sim} H^0(X, \mathcal{O}_X) \to 0$$

$$0 \to H^1(X, \mathcal{O}_X) \to H^2(\mathbb{P}^2, \mathcal{O}(-d)) \to 0$$

$$H^2(\mathbb{P}^2, \mathcal{O}(d)) = 0$$

$$d > -n-1 = -3.$$

$$H^0(X, \mathcal{O}_X) = k$$

$$H^1(X, \mathcal{O}_X) \simeq k^{\frac{(d-1)(d-2)}{2}} = H^2(\mathcal{O}(-d)) = H^0(\mathcal{O}(d-3))$$

rang $\binom{d-3+2}{2} = \binom{d-1}{2}$

Using the results on cohomology of line bundles on $\mathbb{P}^2$, we deduce that $H^0(X, \mathcal{O}_X) \simeq k$ and

$$H^1(X, \mathcal{O}_X) \simeq k^{\binom{d-1}{2}}.$$

The dimension of the cohomology group on the left is the *genus* of the curve X (it will be introduced properly in Chapter 19). So the above can be rephrased as saying *the genus of a plane curve of degree d is* $\frac{1}{2}(d-1)(d-2)$.

$$g = \frac{(d-1)(d-2)}{2}$$

$\therefore$ Dersom $d \geq 3$

$\Rightarrow$ X er ikke rasjonal

(= isomorf til $\mathbb{P}^1$).

14.5 *Extended example: The twisted cubic in* $\mathbb{P}^3$

Let k be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will continue Example 12.23 and consider the twisted cubic curve $C = V(I)$ where $I \subset R$ is the ideal generated by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

!

$$\rightsquigarrow \quad 0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^3} \longrightarrow i_* \mathcal{O}_X \longrightarrow 0$$

Let us by hand compute the group $H^1(X, \mathcal{O}_X)$. Of course we know what the answer should be, since $X \simeq \mathbb{P}^1$, and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Indeed, $S = R/I$ is isomorphic to the third Veronese subring $k[s,t]^{(3)} = k[s^3, s^2t, st^2, t^3]$; the Proj of this ring is $\mathbb{P}^1_k$.

Now, to compute $H^1(X, \mathcal{O}_X)$ on X, it is convenient to relate it to a cohomology group on $\mathbb{P}^3$. We have $H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$ where $i : X \to \mathbb{P}^3$ is the inclusion. The sheaf $i_*\mathcal{O}_X$ fits into the ideal sheaf sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_X \to 0.$$

$H^0(X, \mathcal{O}_X) \ (= k)$

$H^1(X, \mathcal{O}_X) = ?$

where $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^3$. Applying the long exact sequence in cohomology, we get

$$\begin{tikzcd}
 & & 0 & \\
\cdots \arrow[r] & H^1(\mathbb{P}^3, \mathcal{I}) \arrow[r] & H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \arrow[u, equal] \arrow[r] & H^1(\mathbb{P}^3, i_*\mathcal{O}_X) \arrow[dll, out=-10, in=170] \\
 & H^2(\mathbb{P}^3, \mathcal{I}) \arrow[r] & H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \arrow[d, equal] \arrow[r] & \cdots \\
 & & 0 &
\end{tikzcd}$$

$$H^1(\mathcal{O}_X) \cong H^2(\mathbb{P}^3, \mathcal{I}_X)$$

By our description of sheaf cohomology on $\mathbb{P}^3$, $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$, which implies that $H^1(X, \mathcal{O}_X) = H^2(\mathbb{P}^3, \mathcal{I})$. We can compute the latter cohomology group using the exact sequence of Example 12.23:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{I} \to 0.$$

R

$$0 \longrightarrow R(-3)^2 \xrightarrow[\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}]{} R(-2)^3 \longrightarrow I \longrightarrow 0$$

$e_1 \longrightarrow g_1 = x_0 x_2 - x_1^2$

$e_2 \longrightarrow g_2 = x_0 x_3 - x_1 x_3$

$e_3 \longrightarrow g_3 = x_1 x_3 - x_2^2$

$$H^n(\mathbb{P}^n, \mathcal{O}(-d)) = H^0(\mathbb{P}^n, \mathcal{O}(d-n-1))$$
$$\dim = \binom{d-n-1+n}{n} = \underline{\binom{d-1}{n}}$$

Now, taking the long exact sequence we get

$$\begin{tikzcd}
\cdots \arrow[r] & \overset{=0}{H^2(\mathbb{P}^3, \mathcal{O}(-3)^2)} \arrow[r] & H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \arrow[r] & \overset{H^1(X, \mathcal{O}_X) \,=}{H^2(\cancel{X}\,\mathbb{P}^3, \mathcal{I})} \arrow[dll, bend left] \\
& H^3(\mathbb{P}^3, \mathcal{O}(-3)^2) \arrow[r] \arrow[d, phantom, "\|"] & H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \arrow[r] \arrow[d, phantom, "\|"] & H^3(\cancel{X}\,\mathbb{P}^3, \mathcal{I}) \\
& 0\ ! & 0 &
\end{tikzcd}$$

$$H^n(\mathbb{P}^n_A, \mathcal{O}(d)) = 0 \qquad d > -n-1$$

$\rightsquigarrow \quad H^2(\mathbb{P}^3, \mathcal{I}) = 0$

$\rightsquigarrow \quad g = 0.$

Here $H^2(\mathbb{P}^3, \mathcal{O}(-2)) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(-3)) = 0$ by our prevision computations. Hence by exactness, we find $H^2(\mathbb{P}^3, \mathcal{I}) = 0$. It follows that $H^1(X, \mathcal{O}_X) = 0$ also, as expected.

$\mathcal{E}$ lokalt fritt rang r $\Leftrightarrow$ $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^r$ $\mathcal{U} = \{U_i\}$

$\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_r$ split

locally free sheaves

14.6 *Extended example: Non-split ~~vector bundles~~*

A locally free sheaf is said to be *split* if it is isomorphic to a direct sum of invertible sheaves. We have seen several examples of locally free sheaves that are not free, even on affine schemes, but a priori it is not so clear whether these are direct sums of projective modules of rank 1. In this section we will study the sheaf $\mathscr{E}$ from Section 12.9 and show that it is indeed non-split.

Cober ϕ

$R = k[x_0 .. x_n]$

$$0 \to R(-1) \xrightarrow{\phi} R^{n+1} \to M \to 0$$

$$1 \longmapsto (x_0, x_1, \ldots, x_n)$$

Oppgave: Vis at Denne splitter på hver $D_+(x_i)$

tilde

The sheaf $\mathscr{E}$ is the locally free sheaf of rank n on $\mathbb{P}^n_k$ sitting in the exact sequence (12.6)

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-1) \to \mathcal{O}^{n+1}_{\mathbb{P}^n_k} \to \mathscr{E} \to 0.$$

Suppose that $\mathscr{E}$ is not split, i.e., $\mathscr{E}$ is not isomorphic to a direct sum of invertible sheaves. Since $\mathrm{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$ is generated by the class of $\mathcal{O}(1)$, this would mean that $\mathscr{E} \simeq \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_k}(a_n)$ for some integers $a_1, \ldots a_n \in \mathbb{Z}$.

Recall that for $n \geqslant 2$, we have $H^{n-1}(\mathbb{P}^n_k, \mathcal{O}(m)) = 0$ for any $m \in \mathbb{Z}$. So if we could show that $H^{n-1}(\mathbb{P}^n_k, \mathscr{E}) \neq 0$, we would have a contradiction. Actually, it is the case that $H^{n-1}(\mathbb{P}^n_k, \mathscr{E}) = 0$, but we can instead consider $\mathcal{F} = \mathscr{E}(-n)$, which fits into the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \to \mathcal{O}_{\mathbb{P}^n_k}(-n)^{n+1} \to \mathcal{F} \to 0. \qquad \otimes\, \mathcal{O}_{\mathbb{P}^n}(-n)$$

$$\cdots \to H^{n-1}\big(\overset{0}{\cancel{\mathcal{O}_{\mathbb{P}^n}(-n-1)}}\big) \to H^{n-1}\big(\overset{0}{\cancel{\mathcal{O}_{\mathbb{P}^n}(-n)}}\big) \to H^{n-1}(\mathcal{F})$$

$$\to H^n\big(\mathcal{O}_{\mathbb{P}^n}(-n-1)\big) \to H^n\big(\cancel{\mathcal{O}_{\mathbb{P}^n}(-n)}\big) \to \cdots$$

$\|$

k

0

$\rightsquigarrow$ $H^{n-1}(\mathcal{F}) \simeq k$ $\rightsquigarrow$ $\mathcal{F}$ is the <u>split</u>!

Taking the long exact sequence in cohomology, we get

$$\cdots \to H^{n-1}(\mathcal{O}_{\mathbb{P}^n_k}^{n+1}) \to H^{n-1}(\mathcal{F}) \xrightarrow{\delta} H^n(\mathcal{O}_{\mathbb{P}^n_k}(-n-1)) \to H^n(\mathcal{O}_{\mathbb{P}^n_k}^{n+1}) \to \cdots$$

Here the two outer cohomology groups are zero, by Theorem 14.7. Hence, by exactness, we find that $H^{n-1}(\mathbb{P}^n_k, \mathcal{F}(-1)) \simeq H^0(\mathbb{P}^n_k \mathcal{O}_{\mathbb{P}^n_k}) = k$. This implies that $\mathcal{F} = \mathscr{E}(-n)$, and hence $\mathscr{E}$ cannot be a sum of invertible sheaves, and we are done.

14.7 *Extended example: Hyperelliptic curves*

Let us recall the hyperelliptic curves defined in Chapter 3. Let k be a field. For an integer $g \geqslant 1$ we consider the scheme X glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} \text{ and } B = \frac{k[u,v]}{(-v^2 + a_{2g+1}u + \cdots + a_1 u^{2g+1})}$$

As before, we glue $D(x)$ to $D(u)$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

Sprin: $H^1(\mathcal{O}_X) = ?$

Let us compute the cohomology groups of $\mathcal{O}_X$ using Čech cohomology. We will use the affine covering $\mathcal{U} = \{U, V\}$ above. Viewing the first ring as a $k[x]$-module, we can write

$$\frac{k[x,y]}{(-y^2 + a_{2g+1}x^{2g+1} + \cdots + a_1 x)} = k[x] \oplus k[x]y$$

and similarly $B \simeq k[u] \oplus k[u]v$ as a $k[u]$-module.

$$C^0 = \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) = \underline{A} \oplus \underline{B}$$

$$d \downarrow$$

$$C^1 = k[x^{\pm 1}, y] / (y^2 - \cdots)$$

Since $\mathcal{U}$ has only two elements, the Čech complex of $\mathcal{O}_X$ has only two terms, $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ and $\mathcal{O}_X(U \cap V)$ and the differential between them is given by

$$\begin{aligned} d^0 : (k[x] \oplus k[x]y) \oplus \left(k[x^{-1}] \oplus k[x^{-1}]x^{-g-1}y\right) &\to k[x^{\pm 1}] \oplus k[x^{\pm 1}]y \\ (p(x) + q(x)y, r(x^{-1}) + s(x^{-1})x^{-g-1}y) &\mapsto p(x) - r(x^{-1}) + (q(x) - s(x^{-1})x^{-g-1})y \end{aligned}$$

Comparing monomials $x^m y^n$ on each side, we deduce that

$$H^0(X, \mathcal{O}_X) = \operatorname{Ker} d^0 = k$$

and

$$H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^0 = k\{yx^{-1}, yx^{-2}, \dots, yx^{-g}\} \simeq k^g.$$

$\therefore$ genus g

In particular, $\dim_k H^1(X, \mathcal{O}_X) = g$. The latter invariant us usually referred to as the *arithmetic genus* of a curve; we have shown that the hyperelliptic curve X has arithmtic genus g.

For $g = 2$, we get a particularly interesting curve – an irreducible projective curve which cannot be embedded in $\mathbb{P}^2$. Indeed, we showed that for any irreducible curve in $\mathbb{P}^2$ of degree d and the corresponding arithmetic genus equals $\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$. However, there is no integer solution to $\frac{1}{2}(d-1)(d-2) = 2$. This implies the following:

Proposition 14.9 *There exist non-singular projective curves which cannot be embedded in $\mathbb{P}^2$.*

NB: I $\mathbb{P}^1 \times \mathbb{P}^1$ finnes kurver av alle genus!

$\mathrm{Pic}\,\mathbb{P}^2 = \mathbb{Z}$ $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}^2$

Note that we still haven't proved that X is projective. As we have just shown, there is no closed immersion $X \to \mathbb{P}^2$ in general for $g \geqslant 2$. However, it is not hard to see that X can be embedded into the *weighted* projective space $\mathbb{P}(1,1,g+1) = \operatorname{Proj} k[x_0, x_1, w]$ given by the equation

$$w^2 = a_{2g+1} x_0^{2g+1} x_1 + \cdots + a_1 x_0 x_1^{2g+1} \tag{14.4}$$

Note that this makes sense if w has degree $g+1$, but it does not define a subscheme of $\mathbb{P}^2$.