Chapter 2 - Schemes

X affine variety $\rightsquigarrow$ $A = A(X)$ $\rightsquigarrow$ maximal ideals in A
$\leftrightarrow$ points in X

A: commutative ring with 1.

Definition 2.1 *For a ring A we define its* spectrum *as*

$$\operatorname{Spec} A = \{\, \mathfrak{p} \mid \mathfrak{p} \subseteq A \textit{ is a prime ideal} \,\}.$$

The set $\operatorname{Spec} A$ has a topology which generalizes the Zariski topology on a variety, and the definitions are very similar: the closed sets in $\operatorname{Spec} A$ are defined to be those of the form

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq \mathfrak{a} \,\}$$

$\leftarrow$ in particular, $V(\mathfrak{a})$ contains the maximal ideals containing $\mathfrak{a}$.

$\mathfrak{a} \subseteq A$ ideal

Lemma 2.2 *Let A be a ring and assume that $\{\mathfrak{a}_i\}_{i\in I}$ is a family of ideals in A. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in A. Then the following three statements hold true:*

i) $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a})\cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$; ← prime avoidance

ii) $V(\sum_i \mathfrak{a}_i) = \bigcap_i V(\mathfrak{a}_i)$;

iii) $V(A) = \varnothing$ *and* $V(0) = \operatorname{Spec} A$.

⇝ V(a) gives a topology on Spec A.

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}$$

Lemma 2.4 *For two ideals* $\mathfrak{a}, \mathfrak{b} \subset A$ *we have*

i) $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ *if and only if* $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$. *In particular, one has* $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$;

ii) $V(\mathfrak{a}) = \varnothing$ *if and only if* $\mathfrak{a} = A$;

iii) $V(\mathfrak{a}) = \operatorname{Spec} A$ *if and only if* $\mathfrak{a} \subseteq \sqrt{(0)}$.

Distinguished open sets

For an element $f \in A$, we let $D(f)$ be the complement of the closed set $V(f)$, that is,

$$D(f) = \{ \mathfrak{p} \mid f \notin \mathfrak{p} \} = X - V(f).$$

These are clearly open sets and are called *distinguished open sets*.

LEMMA 2.5 *The open sets $D(f)$ form a basis for the topology of* Spec A *when f runs through the elements of A.*

Må vise: Alle åpne U kan skrives som en union av $D(f)$'er.

$U \subseteq \operatorname{Spec} A \Rightarrow U^c = V(I)$ er lukket

$\{f_i\}_{i \in J}$ generatorer for I:

$$U = V(I)^c = V\Big(\sum_{i \in J} (f_i)\Big)^c = \Big(\bigcap_i \underbrace{V(f_i)\Big)^c}_{D(f_i)} = \bigcup_i D(f_i)$$

□

$D(f_i)$ overdekning $\iff$ $1 \in (f_i)_{i \in J}$

Lemma 2.6 *A family* $\{D(f_i)\}$ *forms an open covering of* Spec A *if and only if one may write* $1 = \sum_i a_i f_i$ *with the* a_i*'s being elements from* A *only a finite number of which are non-zero.*

Proof: One has $V(\sum_i (f_i))^c = (\bigcap_i V(f_i))^c = \bigcup_i D(f_i)$, so the open sets $D(f_i)$ constitute a covering if and only if the ideal generated by the f_i's is the whole ring A; that is, if and only if 1 belongs there. But this happens if and only if 1 is a combination of finitely many of the f_i's. ❑

$\mathcal{U}$ overdekning av Spec $A = X$

$\Rightarrow$ $U \in \mathcal{U}$ kan skrives som en union av $D(f)$'er

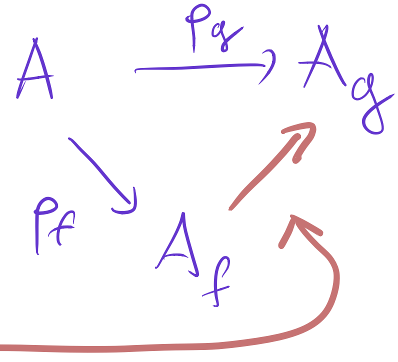
$\Rightarrow$ X dekkes av $D(f)$'er

$\therefore$ Spec A is "quasicompact": Any open cover $\mathcal{U}$ has a finite subcover.

$\Rightarrow$ Kun endelig mange trengs.

LEMMA 2.8 *One has $D(g) \subseteq D(f)$ if and only if $g^n \in (f)$ for a suitable natural number n. In particular, one has $D(f) = D(f^n)$ for all natural numbers n.*

PROOF: The inclusion $D(g) \subseteq D(f)$ holds if and only if $V(f) \subseteq V(g)$, and by Lemma 2.4 on page 45 this is true if and only if $(g) \subseteq \sqrt{(f)}$, i.e., if and only if $g^n \in (f)$ for a suitable n. ❑

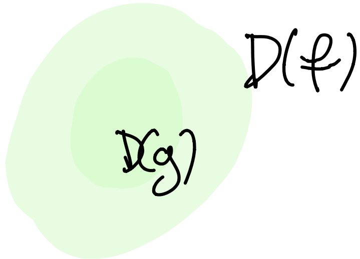


In fact, the inclusion $D(g) \subseteq D(f)$ is equivalent to the condition that the localization map $\rho\colon A \to A_g$ extends to a map $\rho_{fg} : A_f \to A_g$. Indeed, ρ extends if and only if $\rho(f)$, i.e., f regarded as an element in A_g is invertible, which in its turn is equivalent to there being an $b \in A$ and an $m \in \mathbb{N}$ such that $g^m(fb-1) = 0$; or in other words, if and only if $g^m = cf$ for some d and some $m \in \mathbb{N}$.

in other words, if and only if $g^m = cf$ for some d and some $m \in \mathbb{N}$. This enables us to define the localization map by

$D(f) \supseteq D(g) \Rightarrow$

$$\begin{aligned} \rho_{fg} : A_f &\longrightarrow A_g \\ \frac{a}{f^n} &\longmapsto \frac{c^n a}{g^{nm}} \end{aligned}$$



More generally, for an A-module M, we have localization maps

$$\begin{aligned} \rho_{fg} : M_f &\to M_g \\ \frac{x}{f^n} &\mapsto \frac{c^n x}{g^{nm}} \end{aligned}$$

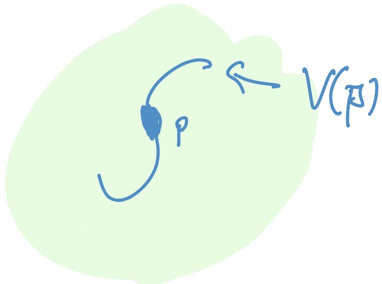
where $x \in M$.

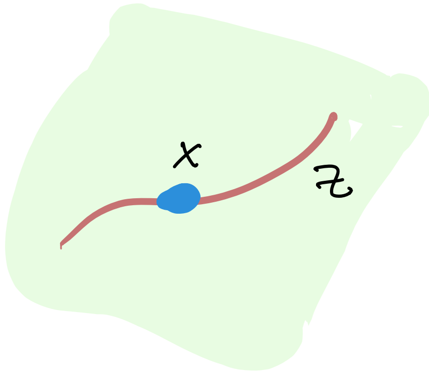
Generic points

det kan hende at $\mathfrak{p} \in \operatorname{Spec} A$ har $V(\mathfrak{p}) = \overline{\mathfrak{p}} \neq \mathfrak{p}$.

PROPOSITION 2.9 *If $\mathfrak{p}$ is a prime ideal of A, the closure $\overline{\{\mathfrak{p}\}}$ of the one-point set $\{\mathfrak{p}\}$ in* Spec A *equals the closed set $V(\mathfrak{p})$.*

PROOF: If the point $\mathfrak{p}$ is contained in a smaller closed set than $V(\mathfrak{p})$, there is an ideal $\mathfrak{a}$ with $\mathfrak{p} \in V(\mathfrak{a}) \subseteq V(\mathfrak{p})$. By lemma 2.4, this implies that $\sqrt{\mathfrak{p}} \subseteq \sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$, from which it follows that $\mathfrak{p} = \sqrt{\mathfrak{a}}$, and hence we conclude that $V(\mathfrak{a}) = V(\mathfrak{p})$. ❑





DEFINITION 2.10 *A point x in a closed subset Z of a topological space X is called a* generic point *of Z if Z is the closure of the singleton $\{x\}$; that is, if $\overline{\{x\}} = Z$,*

Irreducible subsets in Spec A

PROPOSITION 2.11 *Let A be a ring. Then the following statements hold:*

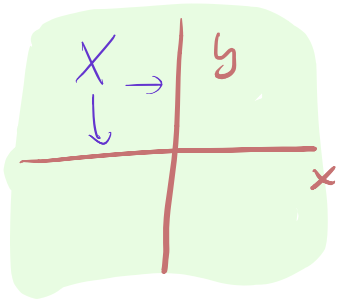
i) A closed subset $Z \subseteq \operatorname{Spec} A$ is irreducible if and only if Z is of the form $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$.

ii) The space $\operatorname{Spec} A$ itself is irreducible if and only if A has just one minimal prime ideal; in other words, if and only if the nilradical $\sqrt{(0)}$ is prime.

PROOF: As the closure of any singleton is irreducible and since we just showed that $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, we know that $V(\mathfrak{p})$ irreducible. For the reverse implication, assume that $V(\mathfrak{a}) \subseteq \operatorname{Spec} A$ is a closed subset. Recall that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}$, so if $\sqrt{\mathfrak{a}}$ is not prime, there are more than one prime involved in the intersection. We may divide them into two different groups thus representing $\sqrt{\mathfrak{a}}$ as the intersection $\sqrt{\mathfrak{a}} = \mathfrak{b} \cap \mathfrak{b}'$ where $\mathfrak{b}$ and $\mathfrak{b}'$ are ideals both different from $\mathfrak{a}$. One concludes that $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{b}')$, so it is not irreducible.

For the second statement it suffices to observe that $\operatorname{Spec} A = V(\sqrt{(0)})$. ❑

A consequence of the lemma is that $\operatorname{Spec} A$ is irreducible whenever A is an integral domain, as in that case (0) is a prime ideal. However, the converse is not true: The ring $A = k[t]/(t^2)$ is not an integral domain, but it has only one prime ideal, (t), so $X = \operatorname{Spec} A$ is just point, hence irreducible.



Example 2.12 The scheme $X = \operatorname{Spec} k[x,y]/(xy)$ is the prime example of a scheme that is connected but not irreducible. The coordinate functions x and y are zero-divisors in the ring $k[x,y]/(xy)$, and their zero-sets $V(x)$ and $V(y)$ show that X has two components. Since these two components intersect at the origin, X is connected. ☆

Functoriality

Let A and B be two rings and let $\phi\colon A \to B$ be a ring homomorphism. The inverse image $\mathfrak{p}^{-1}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subseteq B$ is a prime ideal: that $ab \in \phi^{-1}(\mathfrak{p})$ means that $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, so at least one of $\phi(a)$ or $\phi(b)$ has to lie in $\mathfrak{p}$. Hence sending $\mathfrak{p}$ to $\phi^{-1}(\mathfrak{p})$ gives us a well defined map $\operatorname{Spec} B \to \operatorname{Spec} A$; a map we shall denote by $\operatorname{Spec} \phi$.

$$A \xrightarrow{\phi} B \quad \rightsquigarrow \quad f\colon \operatorname{Spec} B \to \operatorname{Spec} A$$

$$\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$

Lemma 2.13 *Assume that $\phi\colon A \to B$ is a map of rings. Then the induced map between the ring spectra $\operatorname{Spec}\phi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous.*

Proof: We need to show that inverse images of closed sets are closed. So let $\mathfrak{a} \subseteq A$ be an ideal. This follows from series of equalities

$$(\operatorname{Spec}\phi)^{-1}(V(\mathfrak{a})) = \{\, \mathfrak{p} \subseteq B \mid \phi^{-1}(\mathfrak{p}) \supseteq \mathfrak{a} \,\} = \{\, \mathfrak{p} \subseteq B \mid \mathfrak{p} \supseteq \phi(\mathfrak{a}) \,\} = V(\phi(\mathfrak{a})B),$$

which follows because $\mathfrak{p} \supseteq \phi(\mathfrak{a})$ if and only if $\phi^{-1}(\mathfrak{p}) \supseteq \mathfrak{a}$ because $\phi^{-1}(\phi(\mathfrak{a})) \supseteq \mathfrak{a}$. Hence the inverse image $(\operatorname{Spec}\phi)^{-1}(V(\mathfrak{a}))$ is closed. ❑

Example 2.14 *(The spectrum of a quotient, $\operatorname{Spec}(A/\mathfrak{a})$)* If $\mathfrak{a} \subseteq A$ is an ideal, the ring homomorphism $A \to A/\mathfrak{a}$ induces a map

$$f : \operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A.$$

$\operatorname{Im} f = V(a)$

$\bar{\mathfrak{p}} \in \operatorname{Spec}(A/a)$

$\Leftrightarrow \mathfrak{p} \supseteq a$

$\Leftrightarrow \mathfrak{p} \in V(a)$

$$A \xrightarrow{\varphi} A_f \quad \rightsquigarrow \quad \operatorname{Spec} A_f \xhookrightarrow{h} \operatorname{Spec} A$$

$$\operatorname{Im} h = D(f) = \{\mathfrak{p} \mid \mathfrak{p} \not\ni f\}$$

Example 2.15 *(The spectrum of a localization,* Spec A_f*)* For an element $f \in A$, we consider the localization A_f of A in which f is inverted and the corresponding ring homomorphism $A \to A_f$. The prime ideals in the localized ring A_f are in a natural one-to-one correspondence with the prime ideals $\mathfrak{p}$ of A not containing f; in other words, with the complement $D(f) = \operatorname{Spec} A - V(f)$. Thus the induced map $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is a homeomorphism onto the open set $D(f)$ of $\operatorname{Spec} A$. This is an example of an *open immersion*. ☆

$\operatorname{Spec} K = \{(0)\}$

2.2 *Examples*

2.16 *(Fields)* If K is a field, the prime spectrum $\operatorname{Spec} K$ has only one element, corresponding to the zero ideal in K. This also holds true for local rings A with the property that all elements in the maximal ideals are nilpotent, i.e., the radical $\sqrt{(0)}$ of the ring is a maximal ideal. For Noetherian local rings this is equivalent to the ring being an Artinian local ring.

2.18 *(Artinian rings)* More generally, if A is an Artinian ring, then A has only finitely many prime ideals, so $\operatorname{Spec} A$ is a finite set. If A is noetherian, the converse is also true.

$$v : A^\times \longrightarrow \mathbb{Z}$$

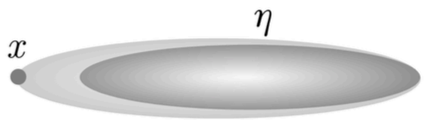
$$v(x+y) \geq \min(v(x), v(y))$$

$$v(xy) = v(x) + v(y)$$

2.19 *(Discrete valuation rings)* Consider a discrete valuation ring A, for example $\mathbb{C}[[t]]$, $k[x]_{(x)}$ or $\mathbb{Z}_{(p)}$). See Appendix A for background on discrete valuation rings). A has only two prime ideals, the maximal ideal $\mathfrak{m}$ and the zero ideal (0). So its prime spectrum $\operatorname{Spec} A$ has just two points, and $\operatorname{Spec} A = \{\eta, x\}$ with x corresponding to the maximal ideal $\mathfrak{m}$ and η corresponding to (0). The point x

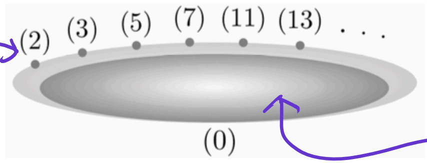
is closed in Spec A, and therefore $\{\eta\} = X - x$ is open. So η is an open point! The point η is the generic point of Spec A; its closure is the whole Spec A.

The open sets of X are $\emptyset, X, \{\eta\}$. In particular Spec A is not Hausdorff, as η is contained in the only open set containing x, the whole space.



The spectrum of a DVR

2.20 *(The spectrum of the integers, Spec $\mathbb{Z}$)* There are two types of prime ideals in $\mathbb{Z}$. There is the zero-ideal and there are the maximal ideals $(p)\mathbb{Z}$, one for each prime p. The latter prime ideals give closed points in Spec $\mathbb{Z}$, however one has $V(0) = \operatorname{Spec}\mathbb{Z}$, so the point corresponding to the zero-ideal is a generic point.



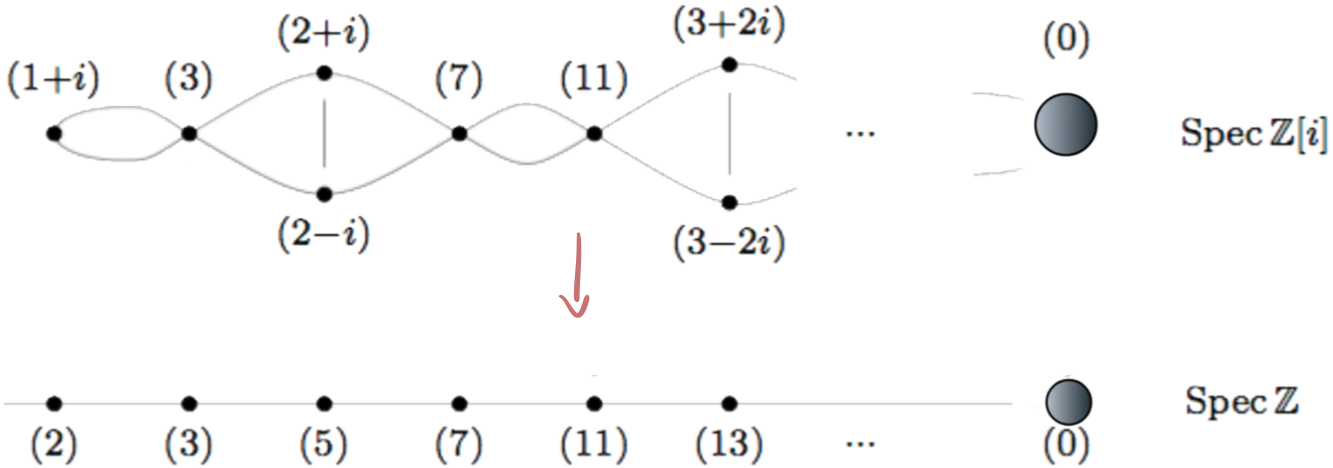
The spectrum Spec $\mathbb{Z}$

The reduction mod p-map $\mathbb{Z} \to \mathbb{F}_p$ induces a map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. The one and only point in $\operatorname{Spec} \mathbb{F}_p$ is sent to the point in $\operatorname{Spec} \mathbb{Z}$ corresponding to the maximal ideal (p). The inclusion $\mathbb{Z} \subseteq \mathbb{Q}$ of the integers in the field of rational numbers induces likewise a map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$, that sends the unique point in $\operatorname{Spec} \mathbb{Q}$ to the *generic* point η of $\operatorname{Spec} \mathbb{Z}$.

$A = \mathbb{Z}[i]$

2.21 *(The spectrum of the Gaussian integers,* $\operatorname{Spec}\mathbb{Z}[i]$*)* The inclusion $\mathbb{Z} \subseteq \mathbb{Z}[i]$ induces a continuous map

$$\phi : \operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}.$$



The spectrum Spec$(\mathbb{Z}[i])$

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}[i] \quad \rightsquigarrow \quad \begin{array}{c} \operatorname{Spec} \mathbb{Z}[i] \\ \downarrow \\ \operatorname{Spec} \mathbb{Z} \end{array}$$

We will study $\operatorname{Spec} \mathbb{Z}[i]$ by studying the fibres (i.e., preimages) of this map. If $p \in \mathbb{Z}$ is a prime, the fibre over $(p)\mathbb{Z}$ consists of those primes that contain $(p)\mathbb{Z}[i]$. These come in three flavours:

i) p stays prime in $\mathbb{Z}[i]$ and the fibre over $(p)\mathbb{Z}$ has one element, namely the prime ideal $(p)\mathbb{Z}[i]$. This happens if and only if $p \equiv 3 \mod 4$;

ii) p splits into a product of two different primes, and the fibre consists of the corresponding two prime ideals. This happens if and only if $p \equiv 1 \mod 4$;

iii) p factors into a product of repeated primes (such a prime is said to 'ramify'). This happens only at the prime (2): note that

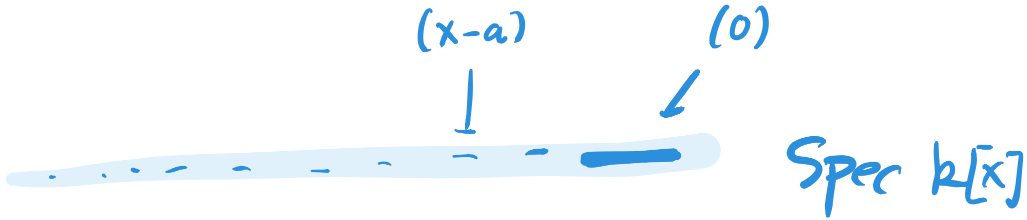
$$(2)\mathbb{Z}[i] = (2i)\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i],$$

which is not radical. So the fibre consists of the single prime $(1+i)\mathbb{Z}[i]$.

Affine n-space

If one lets $A = k[x_1, \ldots, x_n]$ denote the ring of polynomials over k in the variables $x_1, \ldots, x_n$, one knows thanks to Hilbert's Nullstellensatz that the maximal ideals in A stand in a one-to-one correspondence with the points of the affine space $\mathbb{A}^n(k)$; they are all of the form $(x_1 - a_1, \ldots, x_n - a_n)$ with the a_i's being elements in k.

The affine variety $\mathbb{A}^n(k)$ is the subset of the scheme $\mathbb{A}^n_k = \operatorname{Spec} A$ consisting of the closed points; that is, the points in $\operatorname{Spec} A$ corresponding to maximal ideals. The good old Zariski topology on the variety $\mathbb{A}^n(k)$ is the induced topology. Indeed, the closed sets of the induced topology are by definition all

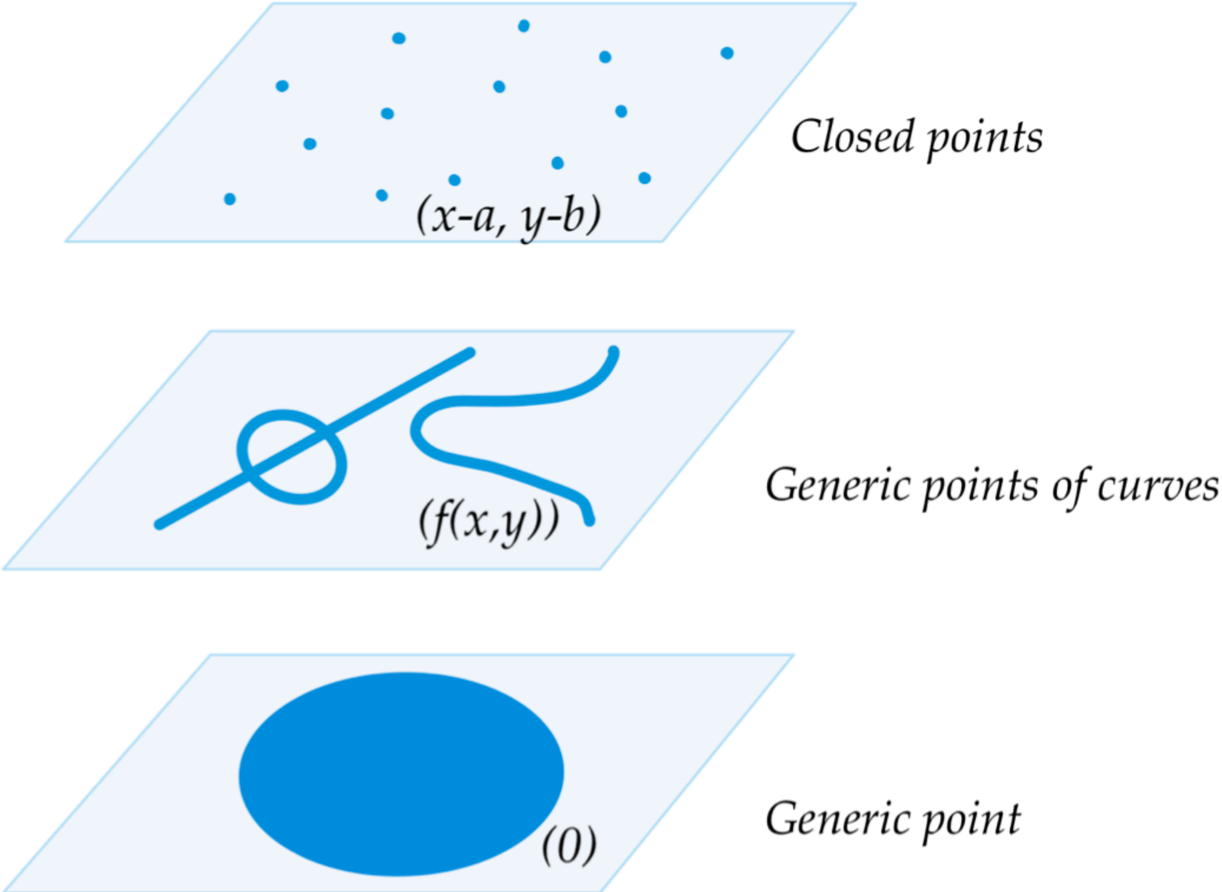


2.22 *(The affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$)* In the polynomial ring $k[x]$ all ideals are principal, and all non-zero prime ideals are maximal. They are of the form $(f(x))$ where $f(x)$ is an irreducible polynomial, hence of the form $(x - a)$ when we assume that k is algebraically closed. There is only one non-closed point in $\operatorname{Spec} k[x]$, the generic point η corresponding to the zero-ideal. The closure $\overline{\{\eta\}}$ is the whole line $\mathbb{A}^1_k$.

2.23 *(The affine plane $\mathbb{A}^2_k = \operatorname{Spec} k[x_1, x_2]$)*

$\mathfrak{p} \subset k[x, y]$

- $\mathfrak{p} = (0)$
- $\mathfrak{p} = (f(x, y))$ f irred.
- $\mathfrak{p} = (x - a, y - b)$ maksimalt



(x-a, y-b)
Closed points
(f(x,y))
Generic points of curves
(0)
Generic point

2.3 *The structure sheaf on* Spec A

We have now come to point where we define the structure sheaf on the topological space Spec A. This is a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ whose stalks all are local rings, so that the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is what one calls a *locally ringed space*.

The two most important properties of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ are the following:

- Sections over distinguished opens: $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$;
- Stalks: $\mathcal{O}_{\operatorname{Spec} A, x} = A_{\mathfrak{p}_x}$.

Definition 2.26 *Let $\mathscr{B}$ be the collection of distinguished open subsets $D(f)$. We define the $\mathscr{B}$-presheaf $\mathcal{O}$ by*

$$\mathcal{O}(D(f)) = A_f,$$

and for $D(g) \subseteq D(f)$ we let the restriction map be localization map $A_f \to A_g$ of (2.2).

Let $S_{D(f)}$ be the multiplicative system $\{s \in A \mid s \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in D(f)\}$. There is a localization map $\tau \colon A_f \to S_{D(f)}^{-1} A$ since $f \in S_{D(f)}$. The following lemma says that the ring $\mathcal{O}(D(f))$ is independent of which ring element f used to define $D(f)$:

Lemma 2.27 *The map τ is an isomorphism, permitting us to identify $A_f = S_{D(f)}^{-1} A$.*

Lemma 2.27 *The map τ is an isomorphism, permitting us to identify $A_f = S_{D(f)}^{-1}A$.*

Proof: The point is that any element $s \in S_{D(f)}$ does not lie in $\mathfrak{p}$ for any $\mathfrak{p} \in D(f)$; in other words, one has $D(f) \subseteq D(s)$. This is equivalent to $\sqrt{(s)} \supset \sqrt{(f)}$, so one may write $f^n = cs$ for some $c \in A$ and $n \in \mathbb{N}$. Assume that $af^{-m} \in A_f$ maps to zero in $S_{D(f)}^{-1}A$. This means that $sa = 0$ for some $s \in S_{D(f)}$. But then $f^n a = csa = 0$, and therefore $a = 0$ in A_f. This shows that the map τ is injective. To see that is surjective, take any as^{-1} in $S_{D(f)}^{-1}A$ and write is as $as^{-1} = ca(cf^n)^{-1} = caf^{-n}$. ❑

Proposition 2.28 *$\mathcal{O}$ is a $\mathcal{B}$-sheaf of rings.*

$$s \longmapsto (s|_{U_i}) \qquad (s_i) \longmapsto (s_i - s_j|_{U_i \cap U_j})$$

$$0 \to \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

$\mathcal{F}$ knippe $\Leftrightarrow$ eksakt

Unraveling the definitions, this can be rephrased as a concrete statement in commutative algebra. We are given a distinguished set $D(f)$ and an open covering $D(f) = \bigcup_{i \in I} D(f_i)$, where we by quasi-compactness may assume that the index set I is finite. Of course then $D(f_i) \subseteq D(f)$, and we have localization maps $\rho_i \colon A_f \to A_{f_i}$ and $\rho_{ij} \colon A_{f_i} \to A_{f_i f_j}$. The statement in the Proposition is then equivalent to the exactness of the following sequence

$$0 \longrightarrow A_f \xrightarrow{\ \alpha\ } \prod_i A_{f_i} \xrightarrow{\ \beta\ } \prod_{i,j} A_{f_i f_j} \qquad (2.4)$$

where $\alpha(a)_i = \rho_i(a)$ and $\beta((a_i))_{i,j} = (\rho_{ij}(a_i) - \rho_{ji}(a_j))$. It is clear that $\alpha \circ \rho = 0$ since $\rho_{ij} \circ \rho_i = \rho_{ji} \circ \rho_j$.

Lemma 2.29 *The sequence (2.4) is exact.*

Proof: We start by observing that we may assume that $A = A_f$ (in other words, that $f = 1$). Indeed, one has $(A_f)_{f_i} = A_{f_i}$ and $(A_f)_{f_i f_j} = A_{f_i f_j}$ since $f_i^{n_i} = h_i f$ for suitable natural numbers n_i.

$$0 \longrightarrow A_f \xrightarrow{\alpha} \prod_i A_{f_i} \xrightarrow{\beta} \prod_{i,j} A_{f_i f_j}$$

Then to the proof: To say that $\alpha(a) = 0$ is to say that a is mapped to zero in each of the localizations A_{f_i}. Hence a power of each f_i kills a; that is, for each index i one has $f_i^{n_i} a = 0$ for an appropriate natural number n_i. The open sets $D(f_i)$ cover $D(f)$, which then is covered by the $D(f_i^{n_i})$ as well. Thus we may write $1 = \sum_i b_i f_i^{n_i}$ for some elements $b_i \in A$, and upon multiplication by a this gives

$$a = \sum_i b_i f_i^{n_i} a = 0.$$

Hence α is injective.

$$0 \longrightarrow A_f \xrightarrow{\alpha} \prod_i A_{f_i} \xrightarrow{\beta} \prod_{i,j} A_{f_i f_j}$$

$a_i \longrightarrow a_i - a_j$

$\exists a$

In down-to-earth terms, the equality $\operatorname{Ker}\beta = \operatorname{Im}\alpha$ means the following: assume given a sequence of elements $a_i \in A_{f_i}$ such that a_i and a_j are mapped to the same element in $A_{f_i f_j}$ for every pair i, j of indices. Then there should be an $a \in A$, such that every a_i is the image of a in A_{f_i}, i.e., $\rho_i(a) = a_i$.

$$0 \longrightarrow A_f \xrightarrow{\alpha} \prod_i A_{f_i} \xrightarrow{\beta} \prod_{i,j} A_{f_i f_j}$$

a_i

Each a_i can be written as $a_i = b_i / f_i^{n_i}$ where $b_i \in A$, and since the indices are finite in number, one may replace n_i with $n = \max_i n_i$. That a_i and a_j induce the same element in the localization $A_{f_i f_j}$ means that we have the equations

$$f_i^N f_j^N (b_i f_j^n - b_j f_i^n) = 0, \tag{2.5}$$

where N *a priori* depends on i and j, but again due to there being only finitely many indices, it can be chosen to work for all. Equation (2.5) gives

$$b_i f_i^N f_j^m - b_j f_j^N f_i^m = 0 \tag{2.6}$$

where $m = N + n$. Putting $b_i' = b_i f_i^N$ we see that a_i equals b_i' / f_i^m in A_{f_i}, and equation (2.6) takes the form

$$b_i' f_j^m - b_j' f_i^m = 0. \tag{2.7}$$

Now $D(f_i^m) = D(f_i)$, and the distinguished open sets $D(f_i^m)$ form an open covering of $\operatorname{Spec} A$. Therefore we may also write $1 = \sum_i c_i f_i^m$. Letting $a = \sum_i c_i b_i'$, we find

$$a f_j^m = \sum_i c_i b_i' f_j^m = \sum_i c_i b_j' f_i^m = b_j' \sum_i c_i f_i^m = b_j',$$

and hence $a = b_j' / f_j^m$ in A_{f_j}. ❑

Definition 2.30 *We let $\mathcal{O}_{\operatorname{Spec} A}$ be the unique sheaf extending the $\mathscr{B}$-sheaf $\mathcal{O}$.*

Explicitly, the sections of $\mathcal{O}_{\operatorname{Spec} A}$ over an open set $U \subset X$, are given by the limit of localizations

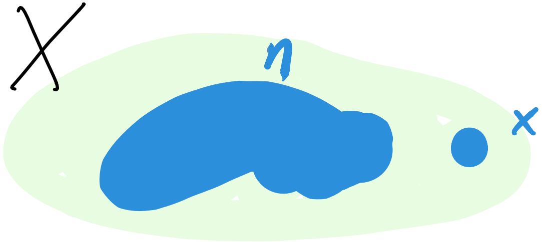
$$\mathcal{O}_X(U) = \varprojlim_{D(f) \subseteq U} A_f. \tag{2.8}$$

Proposition 2.31 *The sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ as defined above is a sheaf of rings satisfying the two paramount properties, namely*

i) *Sections over distinguished opens:* $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$;

ii) *Stalks:* $\mathcal{O}_{\operatorname{Spec} A, x} = A_{\mathfrak{p}_x}$.

In particular, $\Gamma(X, \mathcal{O}_X) = A$.



Example 2.32 Let us continue Example 2.19. In $X = \operatorname{Spec} A$, the spectrum of a DVR, we have three open sets $\varnothing$, η, and X. The structure sheaf takes the following values at these opens:

$$\mathcal{O}_X(\varnothing) = 0, \qquad \mathcal{O}_X(X) = A, \qquad \mathcal{O}_X(\eta) = A_x = K(A),$$

where $K(A)$ denotes the fraction field of A. The stalks are given by $\mathcal{O}_{X,x} = A$ and $\mathcal{O}_{X,\eta} = K(A)$. ☆

Maps between the structure sheaves of two spectra

Previously, we assigned to any map $\phi\colon A \to B$ of rings a continuous map $\phi^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$. We now climb one step in the hierarchy of structures and associate to ϕ a map of sheaves of rings

$$\phi^\sharp\colon \mathcal{O}_{\operatorname{Spec} A} \to \phi_* \mathcal{O}_{\operatorname{Spec} B}.$$

By Proposition 1.29, it suffices to tell what $\phi^\sharp$ should do to the sections over the distinguished open sets $D(f)$. Here everything follows from the following simple lemma:

Lemma 2.33 *Let $\phi\colon A \to B$ be a map of rings and let $f \in A$ be an element. Then $(\phi^*)^{-1}(D(f)) = D(\phi(f))$*

Proof: We have

$$(\phi^*)^{-1}(D(f)) = \{\, \mathfrak{p} \subseteq B \mid f \notin \phi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \subseteq B \mid \phi(f) \notin \mathfrak{p} \,\} = D(\phi(f)).$$

❏

This means that we have the equality $\Gamma(D(f), \phi_* \mathcal{O}_{\operatorname{Spec} B}) = B_{\phi(f)}$, and we know that $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$. The original map of rings $\phi\colon A \to B$ now localizes to a map $A_f \to B_{\phi(f)}$, sending af^{-n} to $\phi(a)\phi(f)^{-n}$, and this shall be the map $\phi^\sharp$ on sections over the distinguished open set $D(f)$.

To prove that $\phi^\sharp$ is well defined, we need to check that it is compatible with the restriction maps among distinguished open sets: indeed, when $D(g) \subseteq D(f)$, we write as usual $g^m = cf$, and the localization map $A_f \to A_g$ will then send af^{-n} to $ac^n g^{-nm}$. One has $\phi(g)^m = \phi(c)\phi(f)$, which makes the diagram below commutative:

$$\begin{array}{ccc} A_f & \longrightarrow & A_g \\ \downarrow & & \downarrow \\ B_{\phi(f)} & \longrightarrow & B_{\phi(g)} \end{array}$$

and this is exactly the required compatibility.

Note by the way, for $\mathfrak{p} \in \operatorname{Spec} B$, with image $\phi^{-1}(\mathfrak{p}) \in \operatorname{Spec} A$, that the stalk map

$$\phi^\sharp_{\mathfrak{p}} : \mathcal{O}_{\operatorname{Spec} A, \phi^{-1}(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}}$$

coincides with the localization $A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$. This is a *map of local rings*, or a *local homomorphism*, in the sense that it the preimage of the maximal ideal of $A_{\phi^{-1}(\mathfrak{p})}$ equals the maximal ideal in $B_{\mathfrak{p}}$.

Towards the definition of a scheme

Definition 2.34 *A* ringed space *is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X. *A* morphism of ringed spaces *is a pair* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *where* $f : X \to Y$ *is continuous, and*

$$f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

is a map of sheaves of rings on Y *(so that* $f^\sharp(U)$ *is a ring homomorphism for each open* $U \subseteq Y$*).*

Definition 2.35 *A* locally ringed space *is a pair* $(X, \mathcal{O}_X)$ *as above, but with the additional requirement that for every* $x \in X$*, the stalk* $\mathcal{O}_{X,x}$ *is a local ring.*

A morphism of locally ringed spaces *is a pair* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *as above, with the additional requirement that for every* $x \in X$*, the map on stalks*

$$f^\sharp_{Y,f(x)} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a map of local rings; that is,

$$(f^\sharp_{Y,f(x)})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$$

where $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ *and* $\mathfrak{m}_{f(x)} \subseteq \mathcal{O}_{Y,f(x)}$ *are the maximal ideals.*

Schemes

Finally, we can give the formal definition of a scheme.

DEFINITION 2.36 ❑ *An* affine scheme *is a locally ringed space* $(X, \mathcal{O}_X)$ *which is isomorphic to* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *for some ring* A.

❑ *A* scheme *is a locally ringed space* $(X, \mathcal{O}_X)$ *that is locally isomorphic to an affine scheme, i.e., there is an open cover* U_i *of* X *such that each* $(U_i, \mathcal{O}_X|_{U_i})$ *is isomorphic to some* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}$.

Sch = category of schemes

AffSch = category of affine schemes

Relative schemes

There is also the notion of *relative schemes* where a base scheme S is chosen. A *scheme over* S is scheme X together with a morphism $f\colon X \to S$, which we call the *structure map* or the *structure morphism*. If two schemes over S are given, say $X \to S$ and $Y \to S$, then a map between them is a map $X \to Y$ compatible with the two structure maps; that is, such that the diagram below is commutative

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \quad \swarrow & \\ & S. & \end{array}$$

$\rightsquigarrow$ category Sch/S

Any ring has a canonical map $\mathbb{Z} \to A$ $\rightsquigarrow$ canonical map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ $\rightsquigarrow$ $AffSch = AffSch/\mathbb{Z}$

$\rightsquigarrow$ $Sch/\mathbb{Z} = Sch$

2.5 *Open immersions and open subschemes*

If X is a scheme and $U \subseteq X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on U, making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is even a scheme, since if X is covered by affines $V_i = \operatorname{Spec} A_i$, then each $U \cap V_i$ is open in V_i, hence can be covered by affine schemes. It follows that there is a canonical scheme structure on U, and we call $(U, \mathcal{O}_X|_U)$ an *open subscheme* of X and say that U has the *induced scheme structure*.. We say that a morphism of schemes $\iota : V \to X$ is an *open immersion* if it is an isomorphism onto an open subscheme of X.

As a special case, consider $V = \operatorname{Spec} A_f$ and the map $\iota : V \to \operatorname{Spec} A = X$, induced by the localization map $A \to A_f$. This is an open immersion onto the open set $U = D(f) \subset X$. Indeed, we saw in Example 2.15 that ι is a homeomorphism onto U, and it follows from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_U$ coincides with the structure sheaf on $\operatorname{Spec} A_f$. A word of

2.6 *Closed immersions and closed subschemes*

If X is a scheme, we would like to define what it means for a closed subset $Z \subset X$ to be a *closed subscheme* of X. The prototypical example of a closed subscheme is the scheme $\operatorname{Spec}(A/I)$, which as we have seen, embeds as the closed subset $V(I)$ of $\operatorname{Spec} A$. However, there may be many ideals that correspond to the same

$V(I)$ of Spec A. However, there may be many ideals that correspond to the same closed subset $V(\mathfrak{a})$ (as $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$). This makes the definition of a closed

$V(I)$ of Spec A. However, there may be many ideals that correspond to the same closed subset $V(\mathfrak{a})$ (as $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$). This makes the definition of a closed subscheme little bit more complicated than the case of open subsets, as we have to specify the locally ringed space structure on Z, and there is no canonical one.

Definition 2.37 i) *A* closed subscheme *of X is given by a closed subset $Z \subseteq X$ and a sheaf of rings $\mathcal{O}_Z$ on Z so such that $(Z, \mathcal{O}_Z)$ is a scheme and $\iota_* \mathcal{O}_Z \simeq \mathcal{O}_X / \mathscr{I}$ for some sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$, where ι denotes the inclusion map.*

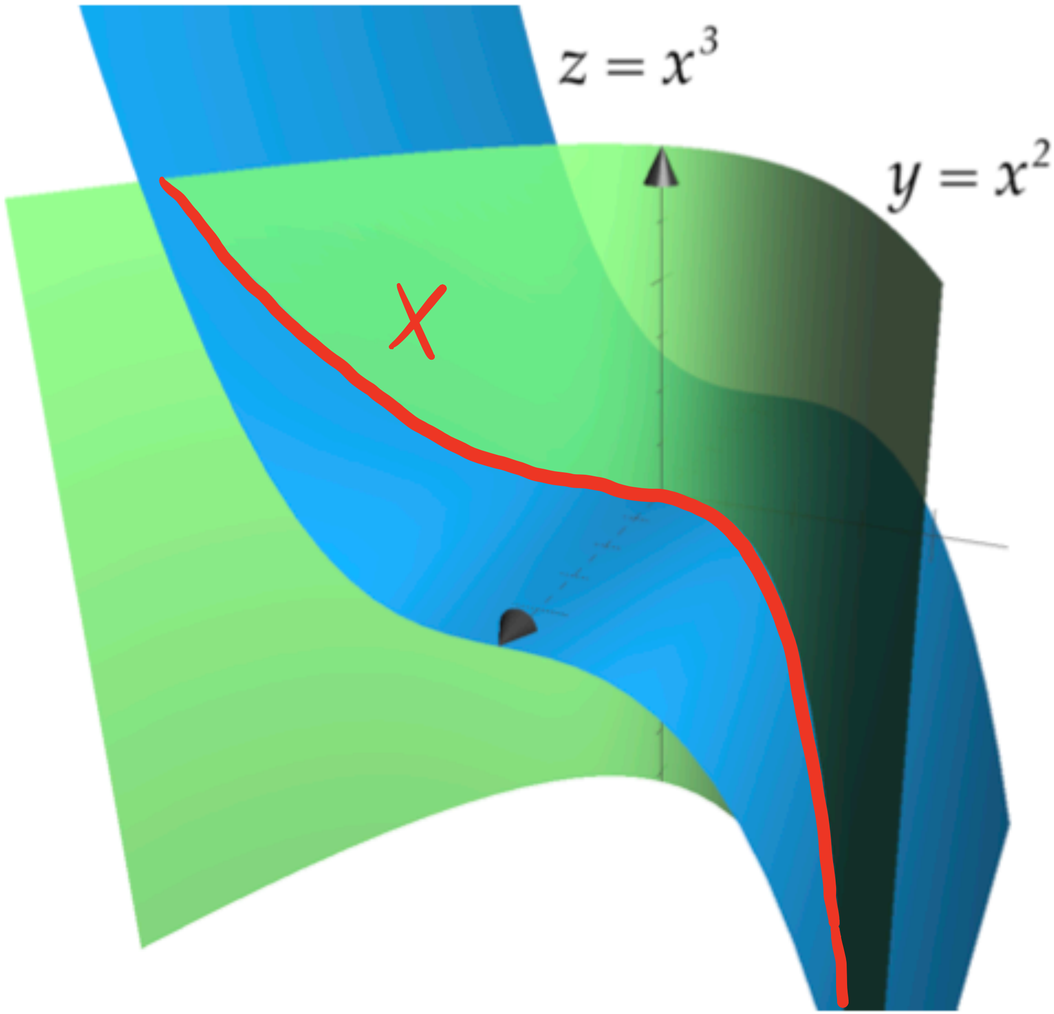
ii) *A morphism $\iota : Z \to X$ is called a* closed immersion *if it induces a homeomorphism of Z onto a closed subset of X, and the sheaf map $i^\sharp : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is surjective.*

Proposition 2.38 *Let* $X = \operatorname{Spec} A$ *be an affine scheme. Then the map* $\mathfrak{a} \mapsto V(\mathfrak{a})$ *induces a one-to-one correspondence between the set of ideals of* A *and the set of closed subschemes of* X*. In particular, any closed subscheme of an affine scheme is also affine.*

EXAMPLE 2.39 Let k be a field. The ring map $\phi : k[x,y,z] \to k[t]$ given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ defines a morphism of schemes

$$f : \mathbb{A}^1_k \to \mathbb{A}^3_k$$

which is a closed immersion. The corresponding closed subscheme is the *twisted cubic curve* $V(I) \subset \mathbb{A}^3_k$ defined by the ideal $I = \operatorname{Ker} \phi = (y - x^2, z - x^3)$.



$z = x^3$
$y = x^2$
X

Example 2.40 Consider the affine 4-space $\mathbb{A}^4_k = \operatorname{Spec} A$, with $A = k[x, y, z, w]$. Then the three ideals

$$I_1 = (x, y), I_2 = (x^2, y) \text{ and } I_3 = (x^2, xy, y^2, xw - yz),$$

give rise to the same closed subset $V(x, y) \subset \mathbb{A}^4_k$, but they give different closed subschemes of $\mathbb{A}^4_k$. ☆

2.7 *Residue fields*

For varieties, we construct the sheaf $\mathcal{O}_X$ from the regular functions which we think of as continuous maps $X \to k$. However, in the world of schemes, we do not have the luxury of having a field k to map into – all we know is that locally $\mathcal{O}_X$ is built from elements of a ring.

We can still define an analogy between the elements f of A and some sort of functions on Spec A. If x is a point in Spec A corresponding to the prime ideal $\mathfrak{p}$, the localization $A_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and one obtains the field $k(\mathfrak{p}) = A_{\mathfrak{p}}/(\mathfrak{p}A_{\mathfrak{p}})$. The element f reduced modulo $\mathfrak{p}$ gives an element $f(x) \in k(\mathfrak{p})$, which may considered as the 'value' of f at x; clearly $f(x) = 0$ if and only if $f \in \mathfrak{p}$.

Definition 2.41 *The field* $k(\mathfrak{p})$ *is called the* residue field *of* Spec A *at* $\mathfrak{p}$.

Note: $V(\mathfrak{a}) = \{x \in \operatorname{Spec} A \mid f(x) = 0 \text{ for all } f \in \mathfrak{a}\}$

This generalizes to arbitrary schemes:

Definition 2.42 *For a scheme X, we can define the* residue field $k(x)$ *at a point* $x \in X$ *as* $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$*, where* $\mathfrak{m}_x$ *is the maximal ideal in* $\mathcal{O}_{X,x}$*.*

If $U \subset X$ is an open set containing x, and $s \in \mathcal{O}_X(U)$ (or if s is an element of $\mathcal{O}_{X,x}$), we let $s(x)$ denote the class of s modulo $\mathfrak{m}_x$ in $k(x)$ – this is the 'value' of s at x.

Note in particular that we may speak of the zero set $V(s) = \{x \in U | s(x) = 0\}$ of the section $s \in O_X(U)$. This is a closed subset of U.

EXAMPLE 2.43 Consider $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$. When k is algebraically closed, there are two types of points, the maximal ideals $(t-a)$ and (0). The residue fields are of the form $k(a) = k[t]_{(t-a)}/(t-a) \simeq k$ and $k(0) = k[t]_{(0)} = k(t)$.

When k is not algebraically closed, we have more interesting residue fields; for instance $\mathfrak{p} = (x^2+1)$ defines a point in $\mathbb{A}^1_{\mathbb{R}}$ with residue field $\mathbb{C}$. In general, a maximal ideal $\mathfrak{m}$ in $k[t]$ is generated by an irreducible polynomial, say $f(t)$, and defines a point in $\mathbb{A}^1_k$ whose residue field is the extension of k obtained by adjoining a root of f. ☆

It is important to note that the 'values' of an element $f \in A$ lie in different fields which might vary with the point. For instance, the element $f = 17 \in \mathbb{Z}$ defines a function on $X = \operatorname{Spec} \mathbb{Z}$. Some of its values are given by

$$f((2)) = \overline{1}, f((3)) = \overline{2}, f((7)) = \overline{3}, f((11)) = \overline{6}, f((17)) = \overline{0}, f(((19)) = \overline{17},$$

and each value has to be interpreted as an element in the appropriate residue field $\mathbb{Z}/p\mathbb{Z}$. Thus we tweak our notion of a 'regular function' on X; they are not maps into some fixed field, but rather maps into the disjoint union $\coprod_{x \in X} k(x)$.

2.8 *R-valued points*

For a scheme X, it makes sense to study morphisms $\operatorname{Spec} R \to X$ from affine schemes into it. We call such morphisms *R-valued points*, and the set of all such will be denoted by $X(R)$.

Example 2.44 Let $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. An R-valued point of $\mathbb{A}^n$ is a morphism $g : \operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$, which determines and is determined by the n-tuple $a_i = g^*(x_i)$ of elements in R. Hence,

$$\mathbb{A}^n(R) = R^n.$$

Now, let $X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I$ where $I = (f_1, \ldots, f_r)$ is an ideal. The set of R-points of X can be found similarly: indeed, any morphism

$$g : \operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I$$

is determined by the n-tuple $a_i = g^*(x_i)$, and those n-tuples that occur are exactly those such that $f \mapsto f(a_1, \ldots, a_n)$ defines a homomorphism

$$\mathbb{Z}[x_1, \ldots, x_n]/I \to R.$$

In other words, the a_i are elements in R which are solutions of the equations $f_1 = \cdots = f_r = 0$. ☆

Example 2.45 *(A conic with no real points)* Let $X = \operatorname{Spec} A$, where A is the real algebra $A = \mathbb{R}[x,y]/(x^2 + y^2 + 1)$. Note that the conic $x^2 + y^2 + 1 = 0$ has no real points, so $X(\mathbb{R}) = \varnothing$. However, A has infinitely many prime ideals. ☆