3.1 *The structure sheaf on* Spec A

We have now come to point where we define the structure sheaf on the topological space $\operatorname{Spec} A$. This is a sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$ whose stalks all are local rings, so that the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is what one calls a *locally ringed space.*

The two most important properties of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ are the following:

- Sections over distinguished opens: $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$;
- Stalks: $\mathcal{O}_{\operatorname{Spec} A, x} = A_{\mathfrak{p}_x}$.

Definition 3.1 *Let $\mathscr{B}$ be the collection of distinguished open subsets $D(f)$. We define the $\mathscr{B}$-presheaf $\mathcal{O}$ by*

$$\mathcal{O}(D(f)) = A_f,$$

and for $D(g) \subset D(f)$ we let the restriction map be localization map $A_f \to A_g$ of (2.2).

Proposition 3.3 *$\mathcal{O}$ is a $\mathscr{B}$-sheaf of rings.*

maps $\rho_i \colon A_f \to A_{f_i}$ and $\rho_{ij} \colon A_{f_i} \to A_{f_i f_j}$. The statement in the Proposition is then equivalent to the exactness of the following sequence

$$0 \longrightarrow A_f \xrightarrow{\ \alpha\ } \prod_i A_{f_i} \xrightarrow{\ \beta\ } \prod_{i,j} A_{f_i f_j} \tag{3.1}$$

where $\alpha(a)_i = \rho_i(a)$ and $\beta((a_i))_{i,j} = (\rho_{ij}(a_i) - \rho_{ji}(a_j))$. It is clear that $\alpha \circ \rho = 0$ since $\rho_{ij} \circ \rho_i = \rho_{ji} \circ \rho_j$.

Lemma 3.4 *The sequence (3.1) is exact.*

Definition 3.5 *We let $\mathcal{O}_{\operatorname{Spec} A}$ be the unique sheaf extending the $\mathscr{B}$-sheaf $\mathcal{O}$.*

Explicitly, the sections of $\mathcal{O}_{\operatorname{Spec} A}$ over an open set $U \subset X$, are given by the inverse limit of localizations

$$\mathcal{O}_X(U) = \varprojlim_{D(f) \subset U} A_f. \tag{3.5}$$

Thus $\mathcal{O}_X(U)$ is an A-module, with universal restriction maps into each of the localizations in the inverse system

$$\begin{tikzcd}
 & A_f \arrow[dr] & & & \\
\cdots \arrow[ur] \arrow[dr] & & A_g \arrow[dr] & & \cdots \\
 & A_{f'} \arrow[ur] & & A_h \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & A_{g'} \arrow[ur] \arrow[dr] & & \ddots \\
 & A_{f''} \arrow[ur] & & \ddots &
\end{tikzcd}$$

Proposition 3.6 *The sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ as defined above is a sheaf of rings satisfying the two paramount properties, namely*

i) *Sections over distinguished opens:* $\Gamma(D(f), \mathcal{O}_{\operatorname{Spec} A}) = A_f$;

ii) *Stalks:* $\mathcal{O}_{\operatorname{Spec} A, x} = A_{\mathfrak{p}_x}$.

In particular, $\Gamma(X, \mathcal{O}_X) = A$.

Proof: We defined $\mathcal{O}$ so that the first property would hold. The second follows from Lemma 1.19. The last statement follows by taking $f = 1$. ❑

ex X = affin varietet med $A(X) = A$.

$\rightsquigarrow \mathcal{O}_X(D(f)) = A(X)_f \qquad \frac{a}{f^m}$

Corollary 3.7 *Let A be an integral domain with fraction field K, and let $X = \operatorname{Spec} A$. Then $\mathcal{O}_X$ is naturally a subsheaf of the constant sheaf K_X, and*

$$\mathcal{O}_X(U) = \left\{ f \in K \;\middle|\; \begin{array}{l} f \text{ can be represented as } g/h \\ \text{where } h(x) \neq 0 \text{ for every } x \in U. \end{array} \right\} \subset K$$

Furthermore, we have

i) $\mathcal{O}_X(D(g)) = \{\, a/g^n \mid f \in A,\, n \geqslant 0 \,\} \subset K$

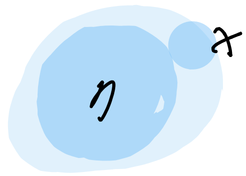
ii) $\mathcal{O}_{X,x} = \{\, f/g \mid f, g \in A, g \notin \mathfrak{p}_x \,\} \subset K$

$g(x) \neq 0$

$\rightsquigarrow$ tilsvarende varietet – til fellet.

ex $A = k[x]_{(x)} \longrightarrow \eta = (0) \quad x = (x) \subset A$

$\emptyset, \quad \eta, \quad X$



Example 3.8 Let us continue Example 2.19. In $X = \operatorname{Spec} A$, the spectrum of a DVR, we have three open sets $\emptyset$, η, and X. The structure sheaf takes the following values at these opens:

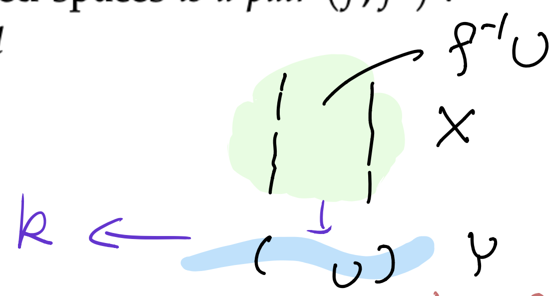
$$\mathcal{O}_X(\emptyset) = 0, \qquad \mathcal{O}_X(X) = A, \qquad \mathcal{O}_X(\eta) = A_x = K(A),$$

where $K(A)$ denotes the fraction field of A. The stalks are given by $\mathcal{O}_{X,x} = A$ and $\mathcal{O}_{X,\eta} = K(A)$. ☆

Definition 3.9 *A* ringed space *is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X*. A* morphism of ringed spaces *is a pair* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *where* $f : X \to Y$ *is continuous, and*

$$f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$$

is a map of sheaves of rings on Y*.*



$U \subseteq Y$

$f_* \mathcal{O}_X (U) = \mathcal{O}_X (f^{-1} U)$

This means that the $f^\sharp(U)$ are ring homomorphisms $f^\sharp(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$, and we require that they commute with the restriction maps:

$$
\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^\sharp(U)"] \arrow[d, "\rho_{U,V}"] & \mathcal{O}_X(f^{-1}U) \arrow[d, "\rho_{f^{-1}U,f^{-1}V}"] & = (f_*\mathcal{O}_X)(U) \\
\mathcal{O}_Y(V) \arrow[r, "f^\sharp(V)"] & \mathcal{O}_Y(V) &
\end{tikzcd}
$$

(tilsvarer f^* for regulære funksjoner $U \to k$)

Definition 3.10 *A* locally ringed space *is a pair* $(X, \mathcal{O}_X)$ *as above, but with the additional requirement that for every* $x \in X$*, the stalk* $\mathcal{O}_{X,x}$ *is a local ring.*

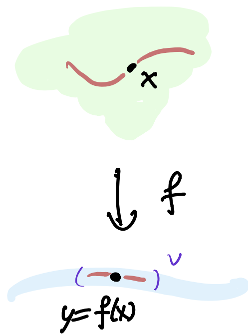
A morphism of locally ringed spaces *is a pair* $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *as above, with the additional requirement that for every* $x \in X$*, the map on stalks*

$$f^\sharp_{Y,f(x)} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a map of local rings; that is,

$$(f^\sharp_{Y,f(x)})^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{f(x)}$$

where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ *and* $\mathfrak{m}_{f(x)} \subset \mathcal{O}_{Y,f(x)}$ *are the maximal ideals.*



$$\mathcal{O}_{\operatorname{Spec} A,\, x} = A_{\mathfrak{p}}$$

$$x \in X \longleftrightarrow \mathfrak{p} \subset A$$

Proposition 3.11 *For a ring A, the pair $(X, \mathcal{O}_X) = (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a locally ringed space. Moreover, for a map of rings $\phi : A \to B$, there is an induced map of locally ringed spaces $(h, h^\sharp) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.*

$$\phi : A \to B \quad \rightsquigarrow \quad \begin{aligned} \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\ \mathfrak{q} &\longmapsto \phi^{-1}(\mathfrak{q}) \end{aligned}$$

PROOF: We defined the structure sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} A$ so that $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ at each point $x = [\mathfrak{p}]$. In particular, the stalks are local rings.

For the second claim, let $\phi\colon A \to B$ be a morphism of rings and let $h\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map given by $h([\mathfrak{p}]) = [\phi^{-1}(\mathfrak{p})]$. We want to associate to ϕ a map of sheaves of rings

$$h^\sharp\colon \mathcal{O}_{\operatorname{Spec} A} \to h_*\mathcal{O}_{\operatorname{Spec} B}.$$

$$\mathcal{O}_{\mathrm{Spec}\,A} \xrightarrow{h^\#} h_* \mathcal{O}_{\mathrm{Spec}\,B} \qquad D(f)$$

$$A_f \longrightarrow B_{\phi(f)}$$

By Proposition 1.53, it suffices to tell what $\phi^\sharp$ should do to the sections over the distinguished open sets $D(f)$. Here we recall Lemma 2.21, which tells us that

$$h^{-1}(D(f)) = D(\phi(f)).$$

This means that we have the equality $\Gamma(D(f), h_*\mathcal{O}_{\mathrm{Spec}\,B}) = B_{\phi(f)}$, and we know that $\Gamma(D(f), \mathcal{O}_{\mathrm{Spec}\,A}) = A_f$. The original map of rings $\phi\colon A \to B$ now localizes

$$
\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\
A_f \arrow[r, "h^\#"] & B_{\phi(f)} \\
\frac{a}{f^n} \arrow[r, mapsto] & \frac{\phi(a)}{\phi(f)^n}
\end{tikzcd}
$$

to a map $A_f \to B_{\phi(f)}$, sending af^{-n} to $\phi(a)\phi(f)^{-n}$, and this shall be the map $h^\sharp$ on sections over the distinguished open set $D(f)$.

To prove that $h^\sharp$ is well defined, we need to check that it is compatible with the restriction maps among distinguished open sets: indeed, when $D(g) \subset D(f)$, we write as usual $g^m = cf$, and the localization map $A_f \to A_g$ will then send af^{-n} to $ac^n g^{-nm}$. One has $\phi(g)^m = \phi(c)\phi(f)$, which makes the diagram below commutative:

$$\begin{tikzcd} A_f \arrow[r] \arrow[d] & A_g \arrow[d] \\ B_{\phi(f)} \arrow[r] & B_{\phi(g)} \end{tikzcd}$$

and this is exactly the required compatibility.

$$\begin{array}{ccc} p & \in & \operatorname{Spec} B \\ \downarrow & & \downarrow h \\ q & \in & \operatorname{Spec} A \end{array}$$

Note that for $[\mathfrak{p}] \in \operatorname{Spec} B$, with image $[\mathfrak{q}] = [\phi^{-1}(\mathfrak{p})] \in \operatorname{Spec} A$, the stalk map

$$h_{\mathfrak{p}}^{\sharp} : \mathcal{O}_{\operatorname{Spec} A, \mathfrak{q}} \to \mathcal{O}_{\operatorname{Spec} B, \mathfrak{p}}$$

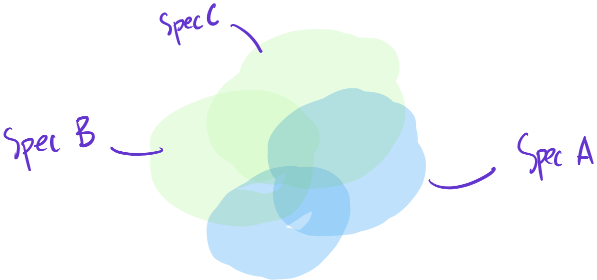
coincides with the localization $A_{\phi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$. Thus the preimage of the maximal ideal of $A_{\phi^{-1}(\mathfrak{p})}$ equals the maximal ideal in $B_{\mathfrak{p}}$, making $h_{\mathfrak{p}}^{\sharp}$ a map of local rings. Hence $(h, h^{\sharp})$ is a morphism of locally ringed spaces. ❑

3.3 *Schemes*

Finally, we can give the formal definition of a scheme.

DEFINITION 3.12 ❑ *An* affine scheme *is a locally ringed space* $(X, \mathcal{O}_X)$ *which is isomorphic to* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *for some ring* A.

❑ *A* scheme *is a locally ringed space* $(X, \mathcal{O}_X)$ *that is locally isomorphic to an affine scheme, i.e., there is an open cover* U_i *of* X *such that each* $(U_i, \mathcal{O}_X|_{U_i})$ *is isomorphic to some affine scheme* $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

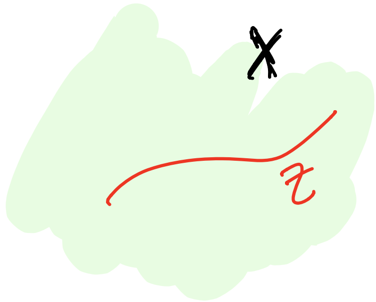


3.4 *Open immersions and open subschemes*

If X is a scheme and $U \subset X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on U, making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is even a scheme, since if X is covered by affines $V_i = \operatorname{Spec} A_i$, then each $U \cap V_i$ is open in V_i, hence can be covered by affine schemes. It follows that there is a canonical scheme structure on U, and we call $(U, \mathcal{O}_X|_U)$ an *open subscheme* of X and say that U has the *induced scheme structure*. We say that a morphism of schemes $\iota : V \to X$ is an *open immersion* if it is an isomorphism onto an open subscheme of X.

$$A \to A_f \rightsquigarrow \operatorname{Spec} A_f \xrightarrow{\sim} D(f) \subset \operatorname{Spec} A$$

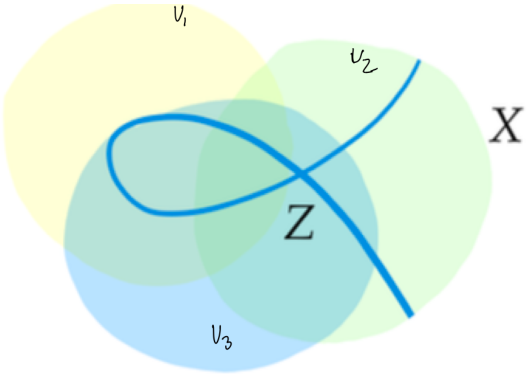
As a special case, consider $V = \operatorname{Spec} A_f$ and the map $\iota : V \to \operatorname{Spec} A = X$, induced by the localization map $A \to A_f$. This is an open immersion onto the open set $U = D(f) \subset X$. Indeed, we saw in Example 2.22 that ι is a homeomorphism onto U, and it follows from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_U$ coincides with the structure sheaf on $\operatorname{Spec} A_f$.



3.5 *Closed immersions and closed subschemes*

If X is a scheme, we would like to define what it means for a closed subset $Z \subset X$ to be a *closed subscheme* of X. This is a little bit more subtle than the case for open subsets, as for a given closed subset $Z \subset X$, there is no canonical locally ringed space structure on Z.

Reason: $\mathcal{O}_X$ does not know about open subsets of Z $\rightsquigarrow$ need to specify a structure sheaf.



$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$

$\rightsquigarrow$ $V(\mathfrak{a})$ husker ikke idealet $\mathfrak{a}$.

The prototypical example of a closed subscheme is $\operatorname{Spec}(A/\mathfrak{a})$, which as we have seen, embeds as the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} A$. Here the scheme structure clear. So we have a clear intuitive picture of what a closed subscheme should be in general: It is a scheme $(Z, \mathcal{O}_Z)$ and with a morphism $i : Z \to X$, so that there is an affine cover $U_i = \operatorname{Spec} A_i$ of X, so that each $i^{-1}(U_i)$ is given by some ideal in A_i (i.e., $i^{-1}(U_i) \simeq \operatorname{Spec}(A_i/\mathfrak{a}_i)$).

$\therefore$ closed subscheme = locally of the form $\operatorname{Spec}(A/\mathfrak{a})$.

DEFINITION 3.14 *i) A* closed subscheme *of X is given by a closed subset $Z \subset X$ and a sheaf of rings $\mathcal{O}_Z$ on Z so such that $(Z, \mathcal{O}_Z)$ is a scheme and $\iota_* \mathcal{O}_Z \simeq \mathcal{O}_X / \mathscr{I}$ for some sheaf of ideals $\mathscr{I} \subset \mathcal{O}_X$, where ι denotes the inclusion map.*

ii) A morphism $\iota : Z \to X$ is called a closed immersion *if it induces a homeomorphism of Z onto a closed subset of X, and the sheaf map $i^\sharp : \mathcal{O}_X \to \iota_* \mathcal{O}_Z$ is surjective.*

Disse er relaterte: $i : Z \longrightarrow X$ lukket imersion

$\Rightarrow \quad J = \ker \iota^\# \quad \rightsquigarrow \quad \mathcal{O}_Z \simeq \mathcal{O}_X / J.$

Note that Z is already required to be a scheme in the definition. Each closed subscheme is determined by a sheaf of ideals $\mathscr{I}$, but not all ideal sheaves $\mathscr{I}$ give rise to a closed subscheme.

Will be proved later:

Proposition 3.15 *Let $X = \operatorname{Spec} A$ be an affine scheme. Then the map $\mathfrak{a} \mapsto \operatorname{Spec}(A/\mathfrak{a})$ induces a one-to-one correspondence between the set of ideals of A and the set of closed subschemes of X. In particular, any closed subscheme of an affine scheme is also affine.*

Example 3.16 Consider the affine 4-space $\mathbb{A}_k^4 = \operatorname{Spec} A$, with $A = k[x, y, z, w]$. Then the three ideals

$$I_1 = (x, y), \quad I_2 = (x^2, y) \text{ and } I_3 = (x^2, xy, y^2, xw - yz),$$

give rise to the same closed subset $V(x, y) \subset \mathbb{A}_k^4$, but they give different closed subschemes of $\mathbb{A}_k^4$. ☆

3.6 *R-valued points*

For a scheme X, it makes sense to study morphisms $\operatorname{Spec} R \to X$ from affine schemes into it. We call such morphisms *R-valued points*, and the set of all such will be denoted by $X(R)$. The jargon here is justified from the following:

important in number theory!

Example 3.17 Let $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$. An R-valued point of $\mathbb{A}^n$ is a morphism $g : \operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]$, which determines and is determined by the n-tuple $a_i = g^{\#}(x_i)$ of elements in R. Hence,

$$\mathbb{A}^n(R) = R^n.$$

$g : \operatorname{Spec} R \to \mathbb{A}^n_{\mathbb{Z}}$

$\Longleftrightarrow \quad g^{\#} : \mathbb{Z}[x_1, \ldots, x_n] \to R$

$\Longleftrightarrow \quad a_i = g^{\#}(x_i) \in R$

Now, let $X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I$ where $I = (f_1, \ldots, f_r)$ is an ideal. The set of R-points of X can be found similarly: indeed, any morphism

$$g : \operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n]/I$$

is determined by the n-tuple $a_i = g^{*}(x_i)$, and those n-tuples that occur are exactly those such that $f \mapsto f(a_1, \ldots, a_n)$ defines a homomorphism

$$\mathbb{Z}[x_1, \ldots, x_n]/I \to R.$$

In other words, the a_i are elements in R which are solutions of the equations $f_1 = \cdots = f_r = 0$. $\star$

Q: Given a scheme X $\rightsquigarrow$ how to describe / study $X(\mathbb{R})$?

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x,y,z]}{(x^n+y^n-z^n)} \qquad X(\mathbb{Q})$$

$$x^2+y^2+1=0$$

EXAMPLE 3.18 *(A conic with no real points)* Let $X = \operatorname{Spec} A$, where A is the real algebra $A = \mathbb{R}[x,y]/(x^2+y^2+1)$. Note that the conic $x^2+y^2+1=0$ has no real points, so $X(\mathbb{R}) = \varnothing$. However, A has infinitely many prime ideals. ☆

$X(\mathbb{C}) \neq 0$:

$\Leftarrow$ $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A$

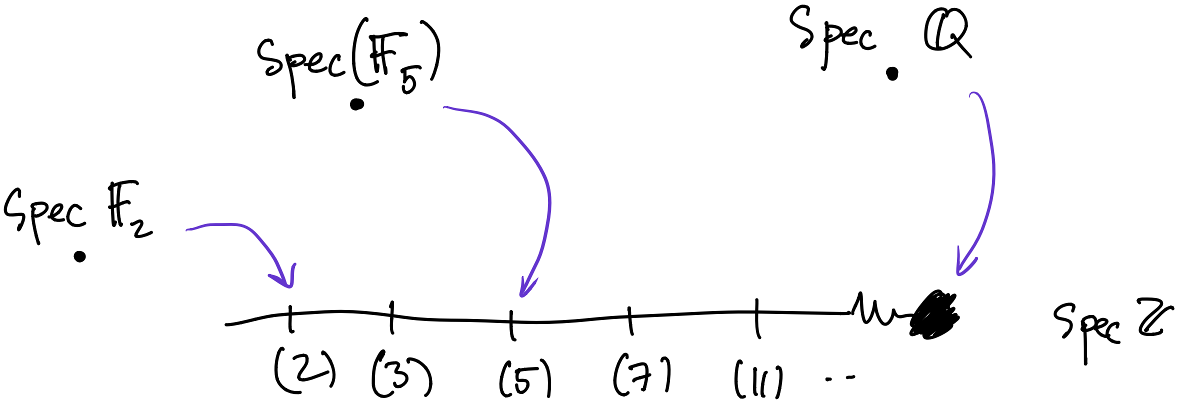
$$\frac{\mathbb{R}[x,y]}{x^2+y^2+1} \xrightarrow{h} \mathbb{C} \quad \text{ring map}$$

$$x \mapsto 0$$
$$y \mapsto i$$

$\rightsquigarrow$ lots of points in $X(\mathbb{C})$

The sets $X(R)$ of points over R are obviously important in number theory, as they naturally generalize the solution set of the polynomials $f_1 = \cdots = f_r = 0$. Of course, even when R is a field, it can be very difficult to describe the set $X(K)$ of K-valued points $\operatorname{Spec} K \to X$, or even determining whether $X(K) \neq \varnothing$.

Proposition 3.19 *Let X be a scheme and let K be a field. Then to give a morphism of schemes $\operatorname{Spec} K \to X$ is equivalent to giving a point $x \in X$ plus an embedding $k(x) \to K$.*



More generally, one may for a fixed scheme S define $X(S)$ to be the set of all morphisms $S \to X$; the so-called *S-valued points* of X. In the example above, we have for any scheme S,

$$\mathbb{A}^n(S) = \mathrm{Hom}_{\mathsf{Sch}}(S, \mathbb{A}^n) = \Gamma(S, \mathcal{O}_S)^n.$$

In fancy terms, this says that $\mathbb{A}^n$ represents the functor taking a scheme to n-tuples of elements of $\Gamma(S, \mathcal{O}_S)$. We shall see a similar functorial characterization of projective space $\mathbb{P}^n$ later in the book.