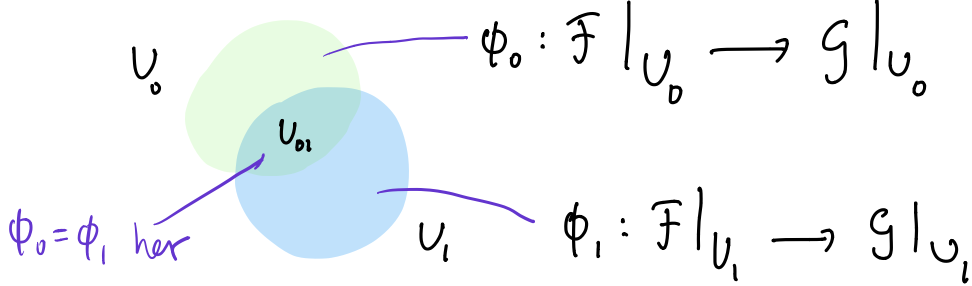
Chapter 4

Gluing and first results on schemes

4.1 Gluing maps of sheaves



We are given two sheaves $\mathcal{F}$ and $\mathcal{G}$ on the topological space X and an open covering $\{U_i\}_{i\in I}$ of X. On each open set U_i, we are given a map of sheaves $\phi_i \colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$, and we assume that the following gluing condition hold on the overlaps:

❑ $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$ ← the maps agree on the overlaps

for all $i, j \in I$. Then we have

Proposition 4.1 (Gluing morphisms of sheaves) *Under the assumptions above, there exists a unique map of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ such that $\phi|_{U_i} = \phi_i$.*

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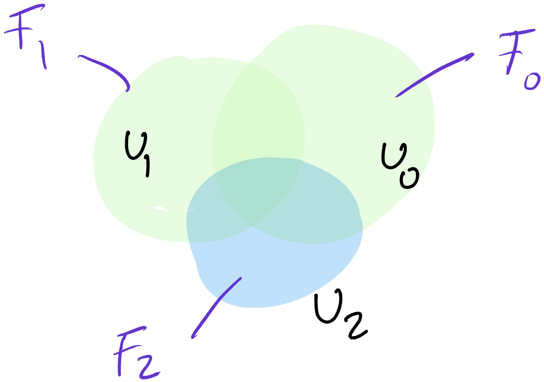
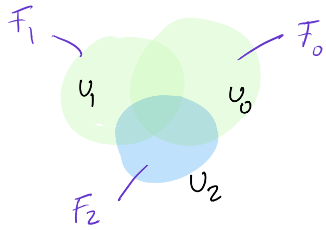
Proof: Take a section $s \in \mathcal{F}(V)$ where $V \subset X$ is open, and let $V_i = U_i \cap V$. Then $\phi_i(s|_{V_i})$ is a well defined element in $\mathcal{G}(V_i)$, and it holds true that $\phi_i(s|_{V_{ij}}) = \phi_j(s|_{V_{ij}})$ by the gluing condition. Hence the sections $\phi_i(s|_{V_i})$'s of the $\mathcal{G}|_{V_{ij}}$'s glue together to a section of $\mathcal{G}$ over V, which we define to be $\phi(s)$. This gives the desired map of sheaves.

The uniqueness also follows: If ϕ and ψ are two morphisms of sheaves so that $\phi(s)|_{U_i} = \psi(s)|_{U_i}$ for all $i \in I$ then $\phi(s) = \psi(s)$, by the Locality axiom for $\mathcal{F}$, and hence $\phi = \psi$. ❏



4.2 *Gluing sheaves*

The setting in this section is a topological space X and an open covering $\{U_i\}_{i \in I}$ of X with a sheaf $\mathcal{F}_i$ on each open subset U_i. We want to "glue" the $\mathcal{F}_i$ together; that is, we search for a global sheaf $\mathcal{F}$ restricting to $\mathcal{F}_i$ on each U_i.



The gluing data consists of isomorphisms $\tau_{ji}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$. The idea is to identify sections of $\mathcal{F}_i|_{U_{ij}}$ with $\mathcal{F}_j|_{U_{ij}}$ using the isomorphisms τ_{ij}. For the gluing process to be possible, the τ_{ij}'s must satisfy the three conditions

- $\tau_{ii} = \mathrm{id}_{\mathcal{F}_i}$, (symmetrisk)
- $\tau_{ji} = \tau_{ij}^{-1}$, (refleksiv)
- $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$, (transitive)

where the last identity takes place where it makes sense; on the triple intersection U_{ijk}.

Proposition 4.2 (Gluing sheaves) *In the setting as above, there exists a sheaf $\mathcal{F}$ on X, unique up to isomorphism, such that there are isomorphisms $v_i \colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$ satisfying $v_j = \tau_{ji} \circ v_i$ over the intersections U_{ij}.*

PROOF: If $V \subset X$ is an open set, we write $V_i = U_i \cap V$ and $V_{ij} = U_{ij} \cap V$. We are going to define the sections of $\mathcal{F}$ over V, and they are of course obtained by gluing sections of the $\mathcal{F}_i$'s along V_i using the isomorphisms τ_{ij}. We define

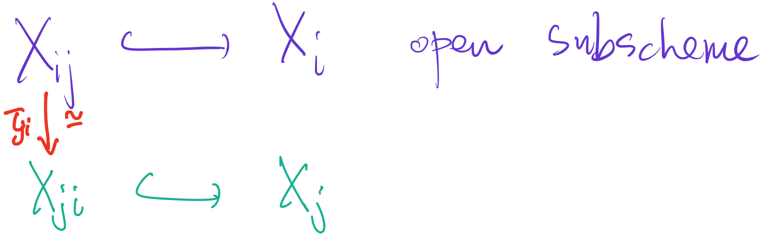
$$\mathcal{F}(V) = \{ (s_i)_{i \in I} \mid \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \} \subset \prod_{i \in I} \mathcal{F}_i(V_i). \tag{4.1}$$

$\rightsquigarrow$ one checks that $\mathcal{F}$ is a sheaf
and $\mathcal{F}|_{V_i} \simeq \mathcal{F}_i$ via the projection $\mathcal{F}(V) \longrightarrow \mathcal{F}_i(V_i)$

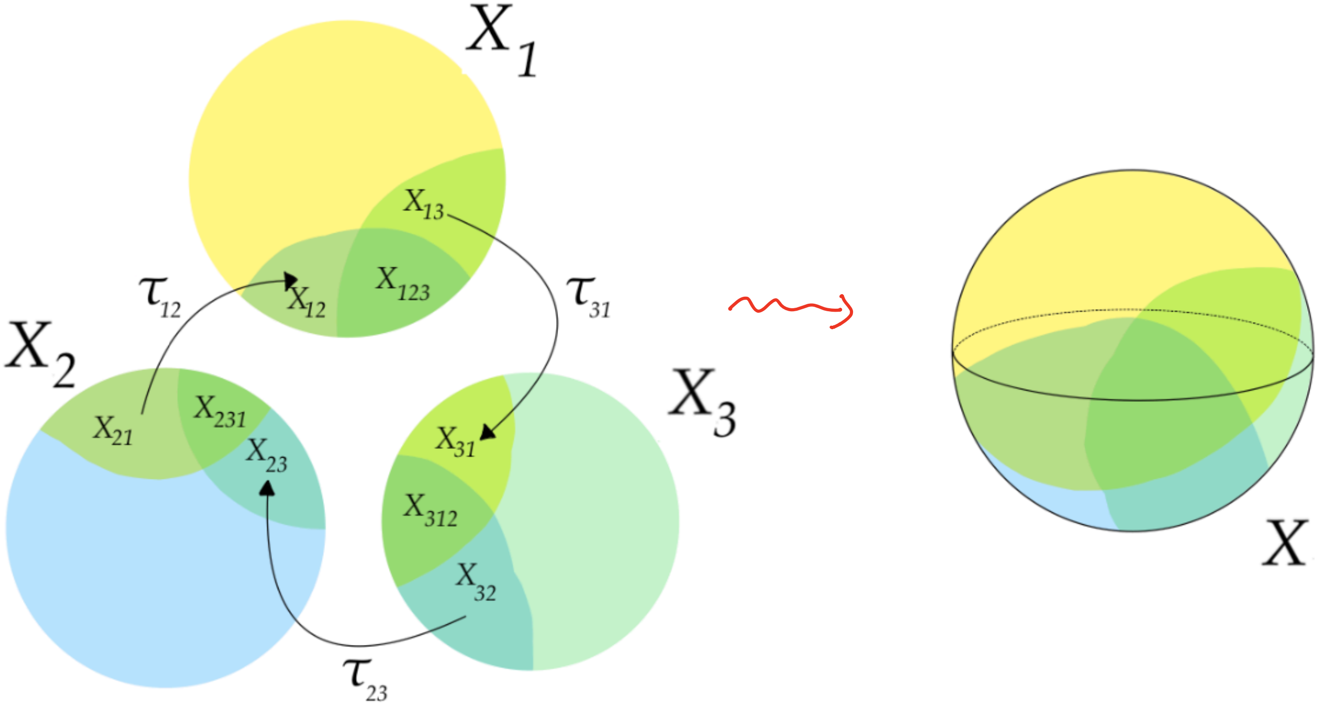
Quick proof using $\mathcal{B}$-sheaves:

EXERCISE 4.2 Let $\{U_i\}_{i\in I}$ be an open cover of X. Let $\mathscr{B}$ be the collection of open sets V so that $V \subset U_i$ for some i. Show that $\mathscr{B}$ is a basis for the topology, and use this to give another proof of Proposition 4.2. ★

4.3 Gluing schemes



In the present setting of 'scheme gluing' we are given a family $\{X_i\}_{i\in I}$ of schemes indexed by a set I. In each of the schemes X_i we are given a collection of open subschemes X_{ij}, where the second index j runs through I. They form the glue lines in the process, i.e., the contacting surfaces that are to be glued together: In the glued scheme they will be identified and will be equal to the intersections of X_i and X_j. The identifications of the different pairs of the X_{ij}'s are encoded by a family of scheme isomorphisms $\tau_{ji}\colon X_{ij} \to X_{ji}$. Furthermore, we let $X_{ijk} = X_{ik} \cap X_{ij}$ – these are the various triple intersections before the gluing has been done – and we have to assume that $\tau_{ij}(X_{ijk}) = X_{jik}$. Notice that X_{ijk} is an open subscheme of X_i.

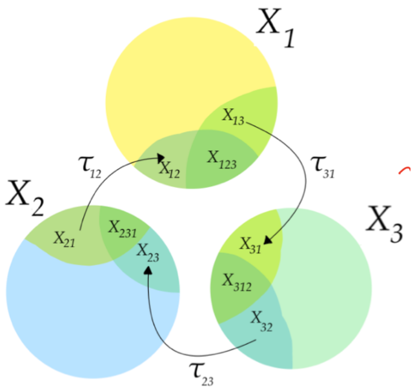


X_1
X_2
X_3
X_{13}
X_{12}
X_{123}
X_{21}
X_{231}
X_{23}
X_{31}
X_{312}
X_{32}
τ_{12}
τ_{31}
τ_{23}
X

The three following gluing conditions, very much alike the ones we saw for sheaves, must be satisfied for the gluing to be possible:

- $\tau_{ii} = \text{id}_{X_i}$. (symmetric)
- $\tau_{ij} = \tau_{ji}^{-1}$. (reflexive)
- The isomorphism τ_{ij} takes X_{ijk} into X_{jik} and one has $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ over X_{ijk}. (transitive)

Proposition 4.3 (Gluing schemes) *Given gluing data X_i, τ_{ij} as above, there exists a scheme X with open immersions $\psi_i \colon X_i \to X$ such that $\psi_i|_{X_{ij}} = \psi_j|_{X_{ji}} \circ \tau_{ji}$, and such that the images $\psi_i(X_i)$ form an open covering of X. Furthermore, one has $\psi_i(X_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$. The scheme X is uniquely characterized by these properties up to a unique isomorphism.*

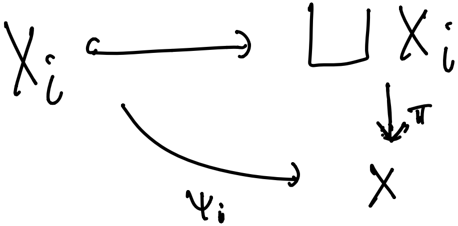


On the level of topological spaces, we start out with the disjoint union $\coprod_i X_i$ and proceed by introducing an equivalence relation on it. We declare two points $x \in X_{ij}$ and $x' \in X_{ji}$ to be equivalent when $x' = \tau_{ji}(x)$. Observe that if the point x does not lie in any X_{ij} with $i \neq j$, we leave it alone, and it will not be equivalent to any other point.

$$\rightsquigarrow \quad X = \bigsqcup X_i \big/ x \sim x'$$

quotient topology using $\pi : \bigsqcup X_i \longrightarrow X$

$U \subseteq X$ open $\Longleftrightarrow$ $\pi^{-1}(U)$ open



Topologically, the maps $\psi_i \colon X_i \to X$ are just the maps induced by the open inclusions of the X_i's in the disjoint union $\coprod_i X_i$. They are clearly injective since a point $x \in X_i$ is never equivalent to another point in X_i. Now, X has the quotient topology so a subset U of X is open if and only if $\pi^{-1}(U)$ is open, and this holds if and only if $\psi_i^{-1}(U) = X_i \cap \pi^{-1}(U)$ is open for all i. In view of the formula

$$\pi^{-1}(\psi_i(U)) = \bigcup_j \tau_{ji}(U \cap X_{ij})$$

τ_{ji} is open

we conclude that each ψ_i is a homeomorphism onto its image.

Notation: $X_i := \psi_i(X_i) \subset X$

$X_{ij} = X_i \cap X_j \subset X$

$X_{ijk} = X_i \cap X_j \cap X_k \subset X$

On X_{ij}, we have the isomorphisms $\tau_{ji}^\sharp \colon \mathcal{O}_{X_j}|_{X_{ij}} \to \mathcal{O}_{X_i}|_{X_{ij}}$; the sheaf maps of the scheme isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$. In view of the third gluing condition $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$, valid on X_{ijk}, we obviously have $\tau_{ki}^\sharp = \tau_{ji}^\sharp \circ \tau_{kj}^\sharp$. The two first gluing conditions translate into $\tau_{ii}^\sharp = \mathrm{id}$ and $\tau_{ji}^\sharp = (\tau_{ij}^\sharp)^{-1}$.

gluing properties needed to apply Proposition 4.2 are satisfied, and we are allowed to glue the different $\mathcal{O}_{X_i}$'s together and thus to equip X with a sheaf of rings. This sheaf of rings restricts to $\mathcal{O}_{X_i}$ on each of the open subsets X_i, and therefore its stalks are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed space that is locally affine, hence a scheme.

ex $U_0 = \operatorname{Spec} k[x] = \mathbb{A}^1_k$

$U_1 = \operatorname{Spec} k[s] = \mathbb{A}^1_k$

open in $\mathbb{A}^1$: $D(x) = \mathbb{A}^1 - 0$

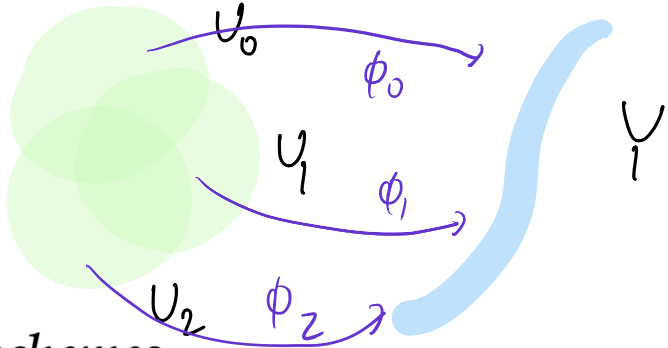
$$k[x] \hookrightarrow k[x^{\pm 1}] = k[x]_x \qquad k[s] \hookrightarrow k[x^{\pm 1}], \quad s \mapsto x^{-1}$$

$\overset{\text{Spec}}{\rightsquigarrow}$

$$\operatorname{Spec} k[x] \hookleftarrow \operatorname{Spec} k[x^{\pm 1}] \hookrightarrow \operatorname{Spec} k[s]$$

$$\mathbb{A}^1 \hookleftarrow D(x) \hookrightarrow \mathbb{A}^1$$

$\rightsquigarrow$ These schemes glue to $\mathbb{P}^1$.



4.4 *Gluing morphisms of schemes*

Suppose we are given schemes X and Y and an open covering $\{U_i\}_{i \in I}$ of X. Assume further that there is given a family of morphisms $\phi_i \colon U_i \to Y$ which match on the intersections $U_{ij} = U_i \cap U_j$. The aim of this paragraph is to show that the ϕ_i's can be glued together to give a morphism $X \to Y$:

$\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$

Proposition 4.4 (Gluing morphisms of schemes) *Given gluing data ϕ_i as above, there exists a unique map of schemes $\phi\colon X \to Y$ such that $\phi|_{U_i} = \phi_i$.*

Proof: Clearly the map on topological spaces is well defined and continuous, so if $U \subset Y$ is an open set, we have to define $\phi^\sharp\colon \Gamma(U, \mathcal{O}_Y) \to \Gamma(U, \phi_* \mathcal{O}_X) = \Gamma(\phi^{-1}U, \mathcal{O}_X)$. So take any section $s \in \mathcal{O}_Y(U)$ over U. This gives sections $t_i = \phi_i^\sharp(s)$ of $\mathcal{O}_X(U_i)$. But since $\phi_i^\sharp$ and $\phi_j^\sharp$ restrict to the same map on U_{ij}, we have $t_i|_{U_{ij}} = t_j|_{U_{ij}}$. The t_i therefore patch together to a section $t \in \mathcal{O}_X(U)$, which is the section we are aiming at: We define $\phi^\sharp(s)$ to be t.

❑

$k[u^{-1}v] \subset k[u,v]_u = k[u^{\pm 1}, v]$

induserev

$k[uv^{-1}] \subset k[u,v]_v = k[u, v^{\pm 1}]$

$$\phi_1 : \operatorname{Spec} k[u,v]_u \longrightarrow \operatorname{Spec} k[u^{-1}v] \subset \mathbb{P}^1$$

$\operatorname{Spec} k[u,v]_u \cong D(u) \subset \mathbb{A}^2$

$$\phi_2 : \operatorname{Spec} k[u,v]_v \longrightarrow \operatorname{Spec} k[uv^{-1}] \subset \mathbb{P}^1$$

$\operatorname{Spec} k[u,v]_v \cong D(v) \subset \mathbb{A}^2$

$$\rightsquigarrow \quad \phi : \mathbb{A}^2 - 0 \longrightarrow \mathbb{P}^1$$

$D(u) \cup D(v) = \mathbb{A}^2 - 0$

$\therefore$ The morphisms ϕ_1 and ϕ_2 glue to the quotient map.

4.5 *The category of affine schemes*

We saw in Chapter 2, that the assignments $A \to \operatorname{Spec} A$ and $\phi \to \operatorname{Spec}(\phi)$ gives a contravariant functor from the category Rings of rings to the category AffSch of affine schemes. There is also a contravariant functor going the other way: taking the global sections of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ gives us the ring A back.

$\varphi : A \longrightarrow B \quad \rightsquigarrow \quad \operatorname{Spec} \varphi : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

Furthermore, a map of affine schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$, comes with a map of sheaves $f^{\#}\colon \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$. Taking global sections gives a ring map

$$A = \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} A, \phi_* \mathcal{O}_{\operatorname{Spec} B}) = B.$$

Let X, Y be two schemes and let $\mathrm{Hom}_{\mathsf{Sch}}(X, Y)$ denote the set of morphisms $f : X \to Y$ between them. We can define a canonical map

$$\Phi : \mathrm{Hom}_{\mathsf{Sch}}(X, Y) \to \mathrm{Hom}_{\mathsf{Rings}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \tag{4.3}$$

which sends $(f, f^\sharp)$ to the map $f^\sharp(Y) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

$$\Phi : \mathrm{Hom}_{\mathsf{Sch}}(X, Y) \to \mathrm{Hom}_{\mathsf{Rings}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

which sends $(f, f^\sharp)$ to the map $f^\sharp(Y) : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.

Proposition 4.5 *If X and Y are affine, the map Φ is bijective.*

Proof: Write $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. By construction, we have $A = \mathcal{O}_Y(Y)$ and $B = \mathcal{O}_X(X)$.

Given a morphism $f : X \to Y$, we let $\phi = \Phi(f) = f^\sharp(Y) : A \to B$ be the corresponding ring map. As we already know, any ring homomorphism induces a morphism of the corresponding ring spectra, so we obtain a morphism of affine schemes $\operatorname{Spec} \phi : X \to Y$. To prove the proposition, we need only prove that this gives an inverse to Φ, i.e., that $\operatorname{Spec} \phi = f$.

Let $x \in X$ be a point, corresponding to the prime ideal $\mathfrak{q} \subset B$, and let $\mathfrak{p} \subset A$ be the prime ideal corresponding to $f(x) \in Y$. We have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{p}} & \longrightarrow & B_{\mathfrak{q}} \end{array}$$

where the vertical maps are the localization maps and the lower map is $f_x^\sharp$. We have $\phi(A \backslash \mathfrak{p}) \subset B \backslash \mathfrak{q}$, so $\phi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$. Now $f_x^\sharp$ is a local homomorphism, so in fact $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$. This means that $\operatorname{Spec} \phi$ induces the same map as f on the underlying topological spaces. Moreover, for each x, the two maps $f_x^\sharp$ and $(\operatorname{Spec} \phi)^\sharp$ coincide with the map $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ above, so also $f^\sharp = (\operatorname{Spec} \phi)^\sharp$ as maps of sheaves. ❑

We have established the following important theorem:

Theorem 4.6 *The two functors* Spec *and* Γ *are mutually inverse and define an equivalence between the categories* Rings *and* AffSch.

In summary, affine schemes X are completely characterized by their rings of global sections $\Gamma(X, \mathcal{O}_X)$, and morphisms between affine schemes $X \to Y$ are in bijective correspondence with ring homomorphisms $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. In particular, a map f between two affine schemes is an isomorphism if and only if the corresponding ring map $f^\sharp$ is an isomorphism.

Proposition 4.7 *Let X be any scheme. Then there is a canonical map of schemes $\psi : X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ inducing the identity on global sections of the structure sheaves.*

Follows by taking id: $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ in:

Theorem 4.8 (Maps into affine schemes) *For any scheme X, the canonical map*

$$\Phi_X : \mathrm{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) \to \mathrm{Hom}_{\mathsf{Rings}}(A, \Gamma(X, \mathcal{O}_X)),$$

given by $(f, f^\sharp) \mapsto f^\#(X)$, is bijective.

PROOF: Let $\{U_i\}$ be an affine covering of X. By the affine schemes case (Theorem 4.6), we know that each Φ_{U_i} is bijective. In particular, from the uniqueness part of Proposition 4.4 we see that Φ_X is injective.

To show that Φ_X is surjective, let $\beta : A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism. By restriction, it induces maps maps $\beta_i : A \to \Gamma(X, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_X)$, and hence morphisms of schemes $f_i : U_i \to \operatorname{Spec} A$. We claim that the f_i's may be glued together to a map $f : X \to \operatorname{Spec} A$. This is a consequence of the fact that the following diagram commutes:

$$\begin{array}{ccccc}
 & & \Gamma(U_i, \mathcal{O}_{U_i}) & & \\
 & \nearrow^{\beta_i} & \uparrow & \searrow & \\
A & \xrightarrow{\beta} & \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(U_i \cap U_j, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X). \\
 & \searrow_{\beta_j} & \downarrow & \nearrow & \\
 & & \Gamma(U_j, \mathcal{O}_{U_j}) & &
\end{array}$$

Indeed, note that for $V \subset U_i \cap U_j$ affine, the diagram implies that the restrictions $f_i|_V$ and $f_j|_V$ induce the same element in $\mathrm{Hom}_{\mathsf{Rings}}(A, \Gamma(V, \mathcal{O}_X))$, and so they are equal on V (using Theorem 4.6). Since this is true for any V, they are equal on all of $U_i \cap U_j$. So by gluing the f_i, we obtain a morphism $f : X \to \operatorname{Spec} A$. We must have $\Phi_X(f) = \beta$ by injectivity of Φ_X and since f maps to $\prod_i \beta_i$ via $\prod_i \Phi_{U_i}$. This completes the proof. ❑

Corollary 4.9 *The canonical map $\psi\colon X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is universal among the maps from X to affine schemes; i.e. any map $\alpha\colon X \to \operatorname{Spec} A$ factors as $\alpha = \alpha' \circ \psi$ for a unique map $\alpha'\colon \operatorname{Spec}\Gamma(X, \mathcal{O}_X) \to \operatorname{Spec} A$.*

Proof: In the theorem above, ψ corresponds to the identity map $\mathrm{id}_{\Gamma(X,\mathcal{O}_X)}$ on the right hand side. The morphism α' is the map of Spec's induced by the ring map $\alpha^\sharp\colon A \to \Gamma(X, \mathcal{O}_X)$. We check that it factors α: The morphism $(\alpha' \circ \psi) : X \to \operatorname{Spec} A$ satisfies $(\alpha' \circ \psi)^\# = \psi^\# \circ \alpha^\# = \alpha^\#$ and hence it coincides with α by the above theorem. ❑

4.6 *The functor from varieties to schemes*

Let k be an algebraically closed field and let V be an algebraic variety over k. We first consider the case where V is affine. Each affine variety has a coordinate ring $A = A(V)$; it is canonically attached to V being the ring of regular functions on V. From $A(V)$ we can build $V^s = \operatorname{Spec} A$, which is an affine scheme whose closed points are in bijection with the points of V (that is, $V^s(k) = V$) according to the Nullstellensatz. Thus the 'new points' correspond to the non-maximal ideals of A.

Moreover, the fundamental theorem of affine varieties tells us that maps $\phi\colon V \to W$ between two affine varieties are in one-one-correspondence with ring maps $\phi^\sharp\colon A(W) \to A(V)$, which exactly parallels our Theorem 4.8. Hence putting $\phi^s = \operatorname{Spec}\phi^\sharp$, we obtain a morphism $\phi^s : V^s \to W^s$ which extends ϕ. As $\phi^\sharp$ is a map of k-algebras, we see that the morphism ϕ^s is a morphism of schemes over $\operatorname{Spec} k$.

Summing up, we have defined a functor $t : \mathsf{AffVar}/\mathsf{k} \to \mathsf{Sch}/\mathsf{k}$. As morphisms of k-varieties $V \to W$ and affine k-schemes $V^s \to W^s$ are both in canonical bijection with k-algebra homomorphisms $A(W) \to A(V)$, the functor t is therefore *fully faithful*, in the sense that

$$\mathrm{Hom}_{\mathsf{Var}/k}(V, W) = \mathrm{Hom}_{\mathsf{Sch}/k}(V^s, W^s).$$

In the general case a variety V has an open cover by affine varieties V_i, and gluing can be performed in both the category of varieties as well as in the categories of schemes, and it is a matter of straightforward checking that this gives a well-defined scheme V^s containing each V_i^s as an open subscheme. The gluing works equally well for morphisms, so we again obtain a functor, which we denote

$$t : \mathsf{Var}/k \to \mathsf{Sch}/k.$$

Once again this functor is fully faithful, in the sense that the induced maps between $\mathrm{Hom}_{\mathsf{Var}/k}(V, W)$ and $\mathrm{Hom}_{\mathsf{Sch}/k}(V^s, W^s)$ are bijective. So two varieties give rise to isomorphic schemes if and only if they are isomorphic as varieties, and the scheme isomorphisms is unambiguously determined by the variety isomorphism. In particular, this tells us that the category of varieties Var/k is equivalent to a full subcategory of Sch/k. We have already seen that t is far from being surjective (Spec $k[x]/x^2$, or: varieties are irreducible), and we shall identify the type of schemes that correspond to its image in Chapter 10.