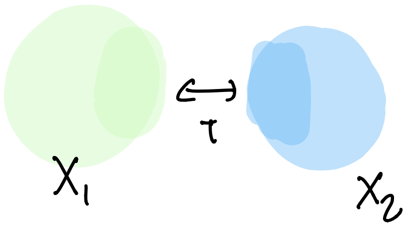
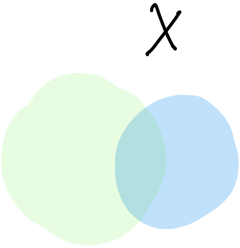
Chapter 5

More examples

5.1 *Gluing two schemes together*





We start out with two schemes X_1 and X_2, with respective open subsets $X_{12} \subset X_1$, $X_{21} \subset X_2$; they are open subschemes so have their canonical induced scheme structures. We also assume that we have an isomorphism $\tau : X_{21} \to X_{12}$. These data allow us to glue together X_1 and X_2 along X_{12} and X_{21}, and thus construct the glued scheme $X = X_1 \cup_\tau X_2$.

On the level of topological spaces, X is obtained from the disjoint union $X_1 \coprod X_2$ by taking the quotient modulo the equivalence relation $x \sim \tau(x)$ for $x \in X_{21} \subset X_2$, and giving X the quotient topology. Moreover, each morphism $X_i \to X$ is an open immersion, allowing us to view X_i as an open subset of X.

The sheaf $\mathcal{O}_X$ is determined by the following exact sequence, where U is any open subset of X.

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X_1}(U \cap X_1) \oplus \mathcal{O}_{X_2}(U \cap X_2) \longrightarrow \mathcal{O}_{X_2}(U \cap X_1 \cap X_2).$$

The components of first map are just the restrictions, and the second sends (r, s) to $t - \tau^\sharp(s)$ (with the appropriate identifications arising from the gluing process).

The main example to keep in mind is when X_1 and X_2 are both affine, say $X_1 = \operatorname{Spec} R$ and $X_2 = \operatorname{Spec} S$, and τ is constructed from a ring isomorphism between localizations

$$\phi : R_u \to S_v$$

for some $u \in R$, $v \in S$. Applying Spec, we get a diagram of schemes

$$\begin{tikzcd}
& \operatorname{Spec} R_u = D(u) \arrow[dl, hook'] & D(v) = \operatorname{Spec} S_v \arrow[l, "\tau"', "\simeq"] \arrow[dr, hook] & \\
\operatorname{Spec} R & & & \operatorname{Spec} S
\end{tikzcd}$$

In this case, the global sections of $\mathcal{O}_X$ can be identified with $\operatorname{Ker}\rho$ in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow R \times S \xrightarrow{\rho} S_v \ , \tag{5.1}$$

where $\rho(r,s) = \iota(s - \phi(r))$, with $\iota\colon S \to S_v$ being the localization map. In other words, elements in $\mathcal{O}_X(X)$ correspond to pairs $(r,s) \in R \times S$ such that $s = \phi(r)$ in the ring S_v.

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A scheme that is not affine

Let k be a field, and $\mathbb{A}^2_k = \operatorname{Spec} A$ where $A = k[u,v]$, and consider $U = \mathbb{A}^2_k - V(u,v)$; this is the affine plane with the closed point corresponding to the origin removed. Since U is an open set of $\mathbb{A}^2_k$, there is a canonical scheme structure on U as described in Section 3.5 in Chapter 3. We claim that U can not be isomorphic to an affine scheme.

<u>ex</u> $\mathbb{A}^2 - V(u) = \mathrm{Spec}\left(k[u,v]_u\right)$ ← affint.

The key point is that the restriction map $\overset{\overset{k[x,y]}{\parallel}}{\Gamma(\mathbb{A}^2_k, \mathcal{O}_{\mathbb{A}^2_k})} \to \Gamma(U, \mathcal{O}_U)$ is an isomorphism, which shows that U can not be an affine scheme; indeed, if that were the case, by Theorem 4.6, the inclusion map $U \to \mathbb{A}^2_k$ would be an isomorphism, which obviously is not true as it is not surjective.

$$U = D(u) \cup D(v)$$

Let us check that the restriction map really is an isomorphism. The two distinguished open sets $D(u) = \operatorname{Spec} A_u$ and $D(v) = \operatorname{Spec} A_v$ form an open affine covering of U, and the exact sequence (5.1) takes the following form:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(U, \mathcal{O}_U) \arrow[r] & A_u \times A_v \arrow[r, "\rho"] & A_{uv} \\
 & A \arrow[u, "i^\sharp"] & &
\end{tikzcd}$$

$$\frac{a}{u^m} \quad \frac{b}{v^n} \to \frac{a}{u^m} - \frac{b}{v^n}$$

where ρ is the difference between the two localization maps; that is, it maps a pair (au^{-m}, bv^{-n}) to $au^{-m} - bv^{-n}$. We have included the restriction map $i^\sharp$ in the diagram, which is the just the map coming from the inclusion map $i\colon U \to \mathbb{A}^2_k$. It sends an element $a \in A$ to the pair $(a/1, a/1)$.

Elements of $\Gamma(U, \mathcal{O}_U)$ correspond to a pairs (au^{-m}, bv^{-n}) in the kernel of ρ. For such a pair we have a relation in $A = k[u, v]$

$$av^n = bu^m,$$

which (since A is a UFD) implies that there is an element $c \in A$ with $a = c\overset{u}{v}^m$ and $b = c\overset{v}{u}^n$; that is, $au^{-m} = bv^{-n}$. This shows that $i^\sharp$ is surjective, and hence an isomorphism since it obviously is injective.

The projective line

alg. geo 1:

$$\mathbb{P}^1 = \mathbb{A}^2 - 0 / k^*$$

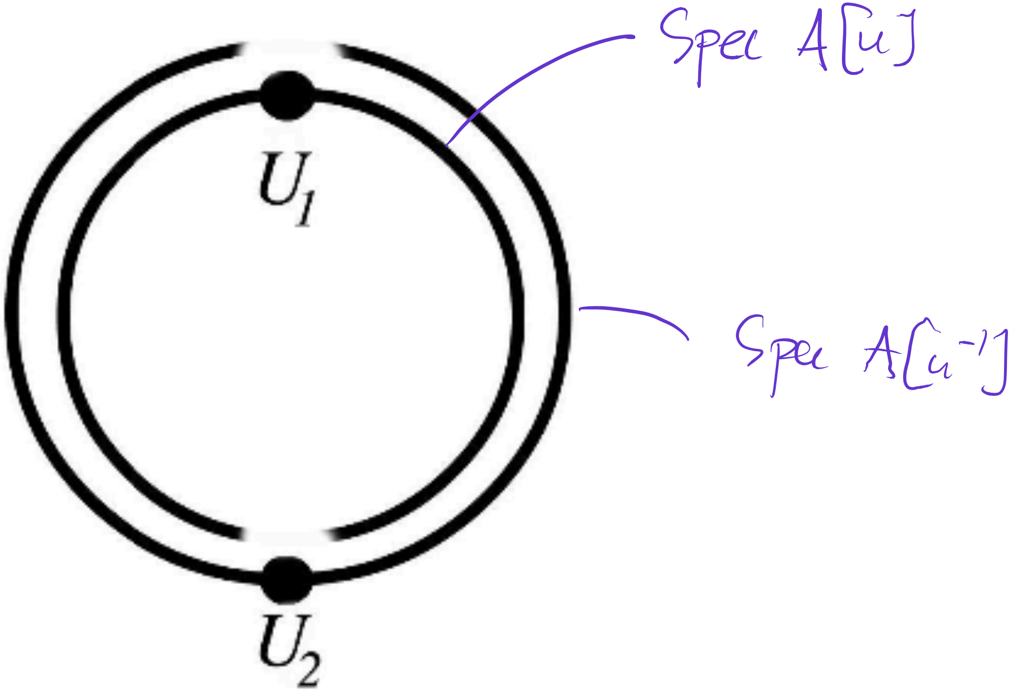
$$= \mathbb{A}^1 \cup \mathbb{A}^1 \text{ over } \mathbb{A}^1 - 0$$

The construction of $\mathbb{CP}^1$ can be vastly generalized, and works in fact over any ring A. Let u be a variable ('the coordinate function at the origin') and let $U_1 = \operatorname{Spec} A[u]$. The inverse u^{-1} is a variable as good as u ('the coordinate at infinity'), and we let $U_2 = \operatorname{Spec} A[u^{-1}]$. Both are copies of the affine line $\mathbb{A}^1_A$ over A.

$$\begin{tikzcd}
\operatorname{Spec} A[u] & \operatorname{Spec} A[u^{-1}] \\
\operatorname{Spec} A[u,u^{-1}] \arrow[u, phantom, sloped, "\subset"] \arrow[r, phantom, "\overset{\tau}{=}"] & \operatorname{Spec} A[u,u^{-1}] \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

Inside U_1 we have the open set $U_{12} = D(u)$ which is canonically isomorphic to the prime spectrum Spec $A[u, u^{-1}]$, the isomorphism coming from the inclusion $A[u] \subset A[u, u^{-1}]$. In the same way, inside U_2 there is the open set $U_{21} = D(u^{-1})$. This is also canonically isomorphic to the spectrum Spec $A[u^{-1}, u]$, the isomorphism being induced by the inclusion $A[u^{-1}] \subset A[u^{-1}, u]$. Hence U_{12} and U_{21} are isomorphic schemes (even canonically), and we may glue U_1 to U_2 along U_{12}. The result is called the *projective line over* A and is denoted by $\mathbb{P}^1_A$.

$$\rightsquigarrow \quad \mathbb{P}^1 = \operatorname{Spec} A[u] \cup_{\tau} \operatorname{Spec} A[u^{-1}].$$



Gluing two affine lines to get $\mathbb{P}^1_A$

Proposition 5.1 *We have* $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}) = A$.

Proof: Since $\mathbb{P}^1_A$ is covered by the two open affines U_1 and U_2, the exact sequence (5.1) above gives us

$$\begin{tikzcd}
\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[r] & \Gamma(U_1, \mathcal{O}_{\mathbb{P}^1_A}) \times \Gamma(U_2, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[r] \arrow[d, "\simeq"] & \Gamma(U_{12}, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[d, "\simeq"] \\
 & A[u] \times A[u^{-1}] \arrow[r, "\rho"'] & A[u, u^{-1}],
\end{tikzcd}$$

$g(u) \quad h(u^{-1}) \longrightarrow g(u) - h(u^{-1})$

$\rightsquigarrow g, h \in A.$

where the map ρ sends a pair $(f(u), g(u^{-1}))$ of polynomials with coefficients in A, one in the variable u and one in u^{-1}, to $g(u^{-1} - f(u)$. We claim that the kernel of ρ equals A; *i.e.* the polynomials f and g must both be constants.

So assume that $f(u) = g(u^{-1})$, and let $f(u) = au^n + \textit{lower terms in } u$, and in a similar way, let $g(u^{-1}) = bu^{-m} + \textit{lower terms in } u^{-1}$, where both $a \neq 0$ and $b \neq 0$, and without loss of generality we may assume that $m \geqslant n$. Now, assume that $m \geqslant 1$. Upon multiplication by u^m we obtain $b + uh(u) = u^m f(u)$ for some polynomial $h(u)$, and putting $u = 0$ we get $b = 0$, which is a contradiction. Hence $m = n = 0$ and we are done. ❑

In particular, the global sections of $\mathcal{O}_X$ of $X = \mathbb{P}^1_{\mathbb{C}}$ is just $\mathbb{C}$.

"Liouville's theorem"

The quotient morphism

In fact, the two examples we have constructed are closely related. In particular, there is a morphism between them:

$$\pi : \mathbb{A}^2_k - V(u, v) \to \mathbb{P}^1_k$$

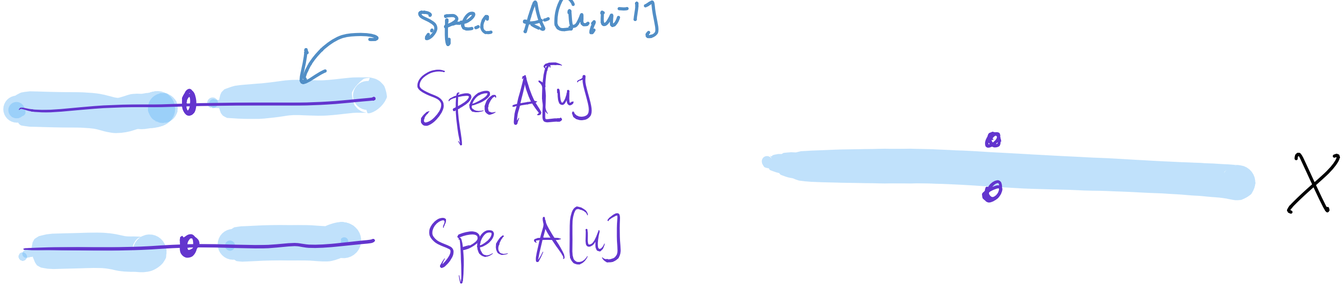
alg. geo ? : $(x, y) \longrightarrow (x : y)$

$$\begin{tikzcd}
D(u) \arrow[rr] \arrow[dr, phantom, sloped, "\subset"] & & \mathbb{A}^1 \arrow[dr, phantom, sloped, "\subset"] & \\
& \mathbb{A}^2 - V(u,v) \arrow[rr] & & \mathbb{P}^1 \\
D(v) \arrow[rr] \arrow[ur, phantom, sloped, "\subset"] & & \mathbb{A}^1 \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}$$

$$\pi : \mathbb{A}_k^2 - V(u,v) \to \mathbb{P}_k^1$$

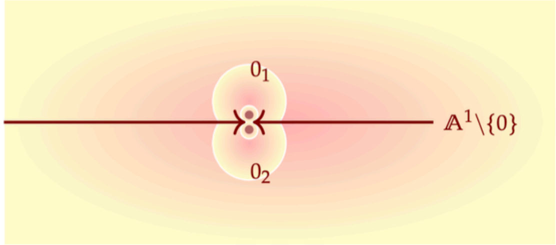
This morphism is constructed by gluing the two morphisms $\phi_1 : \operatorname{Spec} k[u,v]_u \to \operatorname{Spec} k[u^{-1}v]$ and $\phi_2 : \operatorname{Spec} k[u,v]_v \to \operatorname{Spec} k[uv^{-1}]$ induced by the inclusions $k[u^{-1}v] \subset k[u,v]_u$ and $k[uv^{-1}] \subset k[u,v]_v$. The gluing condition is satisfied because of the following diagram:

$$\begin{tikzcd}
k[u,v]_u \arrow[dr, hook] & k[u^{-1}v] \arrow[l, hook'] \arrow[dr, hook] & \\
& k[u,v]_{uv} & k[u^{-1}v, uv^{-1}] \arrow[l, hook'] \\
k[u,v]_v \arrow[ur, hook] & k[uv^{-1}] \arrow[l, hook'] \arrow[ur, hook] &
\end{tikzcd}$$



The affine line with a doubled origin

Keeping the notation from the previous example, we now glue together two copies of the affine line $\mathbb{A}^1_A = \operatorname{Spec} A[u]$ along their common open subset $X_{12} = \operatorname{Spec} A[u, u^{-1}]$. In the notation of the beginning of the chapter, we put $R = S = A[u]$ and use the identity morphism $\phi : A[u, u^{-1}] \to A[u, u^{-1}]$ over the intersection. The resulting scheme contains two $\mathbb{A}^1$'s which overlap outside the origin. But since the gluing does nothing over the origins of each $\mathbb{A}^1$, there are *two* points in X that replace the origin: X is sometimes called the *affine line with two origins*.



This scheme is not affine either: using the sheaf axiom sequence as before, we find that $\Gamma(X, \mathcal{O}_X) \simeq \Gamma(X_1, \mathcal{O}_X) = A[u]$ by the restriction map. However, the map $X_1 = \operatorname{Spec} A[u] \to X$ is not an isomorphism (it is not surjective, since its image misses one of the two origins).

Hyperelliptic curves

Let k be a field and consider the two affine schemes $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, where

$$A = \frac{k[x,y]}{(y^2 - a_{2g+1}x^{2g+1} - \cdots - a_1 x)} \text{ and } B = \frac{k[u,v]}{(v^2 - a_{2g+1}u - \cdots - a_1 u^{2g+1})}$$

with scalars $a_1, \ldots, a_{2g+1} \in k$. The two distinguished open sets $D(x) = \operatorname{Spec} A_x$

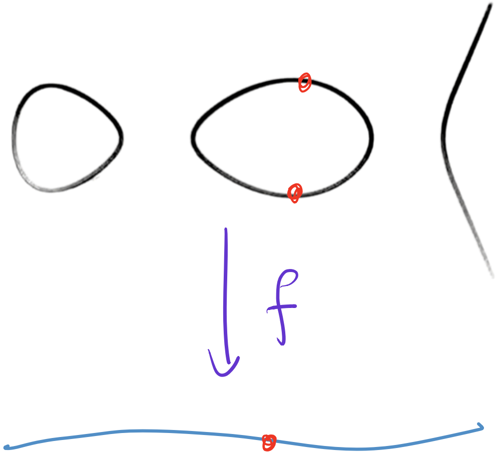
with scalars $a_1, \ldots, a_{2g+1} \in k$. The two distinguished open sets $D(x) = \operatorname{Spec} A_x$ and $D(u) = \operatorname{Spec} B_u$ are isomorphic: the assignments $\phi(u) = x^{-1}$ and $\phi(v) = x^{-g-1}y$ give an isomorphism $\phi\colon B_u \to A_x$. It is well defined as the little calculation

$$\begin{aligned} v^2 - a_{2g+1}u - \cdots - a_1u^{2g+1} &= y^2x^{-2g-2} - a_{2g+1}x^{-1} - \cdots - a_1x^{-2g+1} \\ &= x^{-2g-2}(y^2 - a_{2g+1}x^{2g+1} \cdots - a_1x), \end{aligned}$$

shows that the defining ideal for B_u maps into the one defining A_x, and one verifies effortlessly that the inverse homomorphism is given as $x \mapsto u^{-1}$ and $y \mapsto vu^{-g-1}$. We can thus glue X_1 and X_2 together along the open subsets $D(x)$ and $D(u)$.

$y^2 = x^3 + x$

The resulting scheme X is what is called a *hyperelliptic curve* or a *double cover of* $\mathbb{P}^1_k$. In the case $g = 1$, X is an example of an *elliptic curve*. Here is an illustration of the real points of one of the affine charts for $g = 2$:



The term ‘double cover’ comes from the fact that X admits a morphism $f : X \to \mathbb{P}^1_k$ whose general fibre $f^{-1}(q)$ consists of two points, when k is algebraically closed and of characteristic different from two. That there is such

$\operatorname{Spec} A \longrightarrow \operatorname{Spec} k[x]$

$\operatorname{Spec} B \to \operatorname{Spec} k[u]$

$u = x^{-1}$

a map, follows from the Gluing Lemma for morphisms (Proposition 4.4 on page 84): we have two natural inclusions $k[x] \subset A$ and $k[u] \subset B$, and the "gluing diagram"

$$\begin{tikzcd}
k[u] \arrow[r, "u \mapsto x^{-1}"] \arrow[d, hook] & k[x] \arrow[d, hook] \\
B_u \arrow[r, "\phi"] & A_x
\end{tikzcd}$$

commutes, so the inclusions match on the overlap and patch together to the desired map $X \to \mathbb{P}^1_k$. Note that the correspondence $u \mapsto x^{-1}$ gives the standard construction of $\mathbb{P}^1_k$ by gluing together the two affine lines $\operatorname{Spec} k[x]$ and $\operatorname{Spec} k[u]$.

The blow-up of the affine plane

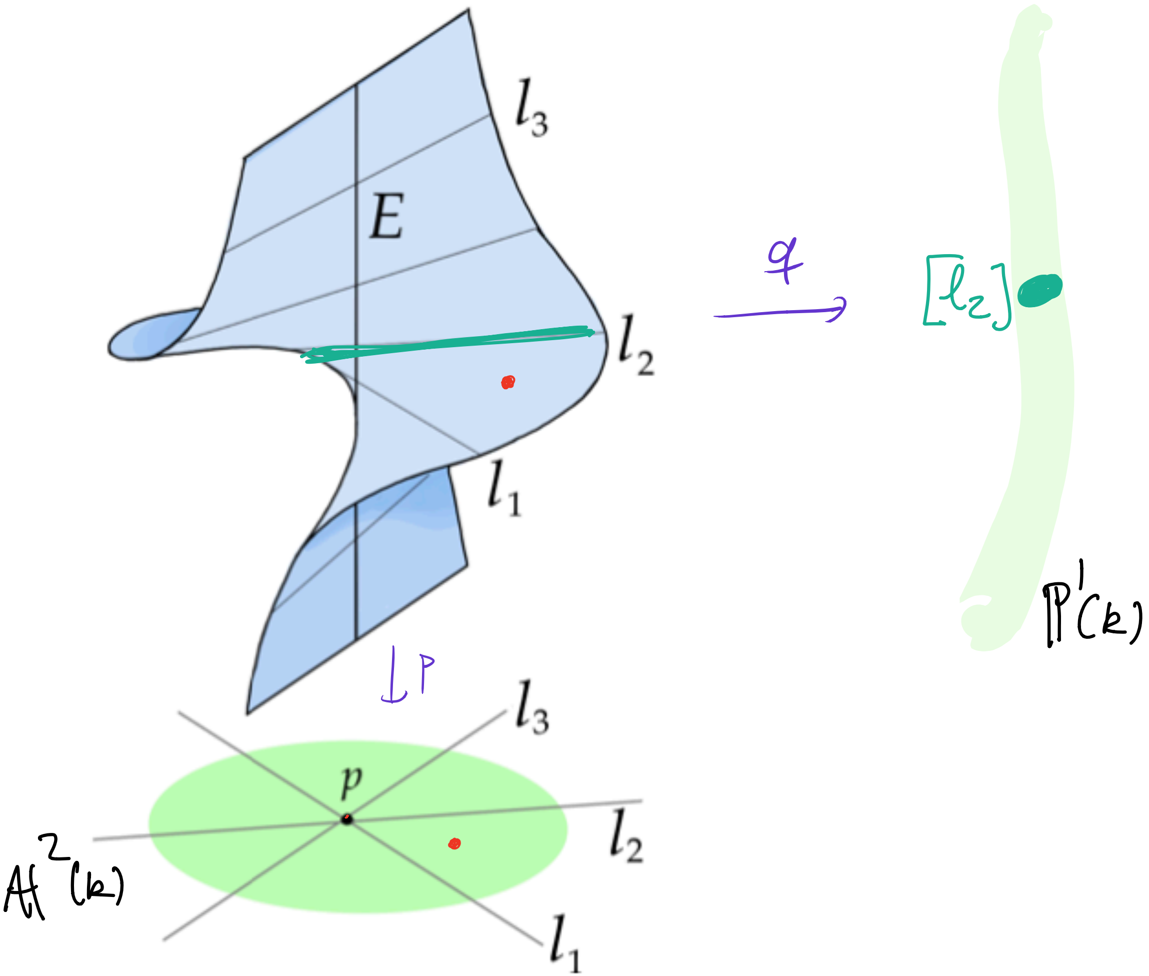
In this section, we will construct the *blow-up of* $\mathbb{A}^2$ *at the origin*, by gluing together two affine schemes. We begin by recalling the classical construction for varieties. To be precise, we write $\mathbb{A}^2(k)$ for the variety, and $\mathbb{A}^2_k$ for the scheme, etc.

THE BLOW-UP AS A VARIETY: Let k be an algebraically closed field, and consider the affine plane $\mathbb{A}^2(k)$. There is a rational map $f : \mathbb{A}^2(k) \dashrightarrow \mathbb{P}^1(k)$ that sends a point (x, y) to the point $(x : y)$ (in homogeneous coordinates on $\mathbb{P}^1_k$). This map is not defined at the origin $p = (0, 0)$, but we can still associate with it the closure X in $\mathbb{A}^2(k) \times \mathbb{P}^1(k)$ of its graph (which lies in $\mathbb{A}^2(k) - (0,0) \times \mathbb{P}^1(k)$).

When describing the graph in detail, it is better to use homogenous coordinates $(s : t)$ on $\mathbb{P}^1(k)$. If the coordinate $t \neq 0$, it holds that $(s : t) = (st^{-1}, 1)$, so the part of the graph where $y \neq 0$, is given by the equation $xy^{-1} = st^{-1}$; in other words, by $xt - ys = 0$, and the same relation gives the part where $x \neq 0$. Hence X is defined in $\mathbb{A}(k)^2 \times \mathbb{P}^1(k)$ by the single equation

$$\mathrm{X} = V(xt - ys) \subset \mathbb{A}^2(k) \times \mathbb{P}^1(k).$$

We also have two projection maps $p : X \to \mathbb{A}^2(k)$ and $q : X \to \mathbb{P}^1(k)$. The situation is depicted in Figure 5.1.



l_3
E
l_2
l_1
q
$[l_2]$
$\mathbb{P}^1(k)$
$\downarrow p$
l_3
p
l_2
$\mathbb{A}^2(k)$
l_1

Let us analyze the fibres of these two maps. The fibres of p are easy to describe. If $(x, y) \in \mathbb{A}^2(k)$ is not the origin, then $p^{-1}(x, y)$ consists of a single point; the equation $xt = ys$ allows us to determine the point $(s : t)$ uniquely since either $x \neq 0$ or $y \neq 0$. However, when $(x, y) = (0, 0)$, any choices of s and t satisfy the equation, so $p^{-1}(0, 0) = \{(0, 0)\} \times \mathbb{P}^1(k)$. In particular, this inverse image is one-dimensional; it is called the *exceptional divisor* of X, and is frequently denoted by E.

Similarly, if $(s:t) \in \mathbb{P}^1(k)$ is a point, the the fibre

$$q^{-1}(s:t) = \{(x,y) \times (s:t) \,|\, xt = ys\} \subset \mathbb{A}(k)^2 \times \{(s:t)\}$$

is a line in $\mathbb{A}^2(k)$. The map q is an example of a *line bundle*; all of its fibres are affine lines $\mathbb{A}^1(k)$'s. We will see these again later on in the book.

The standard covering of $\mathbb{P}^1(k)$ as a union of two $\mathbb{A}^1(k)$'s gives an affine cover of X: If $U \subset \mathbb{P}^1(k)$ is the open set where $s \neq 0$, we can normalize so that $s = 1$, and the equation $xt = sy$ gives $y = tx$. Hence x and t may serve as affine coordinates on $q^{-1}(U)$, and $q^{-1}(U) \simeq \mathbb{A}^2(k)$. In these coordinates, the morphism $p : X \to \mathbb{A}^2_k$ restricts to the map $\mathbb{A}^2(k) \to \mathbb{A}^2(k)$ given by $(x, t) \mapsto (x, xt)$. Similarly, $q^{-1}(V) = \mathbb{A}^2(k)$ with affine coordinates y and s, and the map p is given here as $(y, s) \mapsto (sy, y)$.

THE BLOW-UP AS A SCHEME: From the above discussion, we can define the scheme-analogue of the blow-up of $\mathbb{A}^2_k$ at a point. We will define this as a scheme over $\mathbb{Z}$, rather than over a field k (we get a blow-up of $\mathbb{A}^2_A$ for any ring A by tenzoring everything below by A). Also, in addition to the scheme X, we also want a morphisms of schemes $p : X \to \mathbb{A}^2$ and $q \colon X \to \mathbb{P}^1$ having similar properties to the morphisms in the example above.

Consider the affine plane $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x, y]$. The prime ideal $\mathfrak{p} = (x, y) \subset \mathbb{Z}[x, y]$ corresponds to the closed point p corresponding to the origin $\mathbb{A}^2_k$ in the analogy with situation above. Consider the diagram

$$\begin{tikzcd}
 & \mathbb{Z}[x,y] \arrow[dl, "{x \mapsto x,\ y \mapsto xt}"'] \arrow[dr, "{x \mapsto ys,\ y \mapsto y}"] & \\
\mathbb{Z}[x,t] \arrow[dr, "\text{localising}"] & \circlearrowright & \mathbb{Z}[y,s] \arrow[dl, "\text{localising}"'] \\
 & R = \mathbb{Z}[x,y,s,t]/(xt-y, st-1) &
\end{tikzcd}$$

Here the diagonal maps on the top are given by $x \mapsto x$ and $y \mapsto xt$ respectively $y \mapsto y$ and $x \mapsto ys$.

Note that the ring R is isomorphic to $\mathbb{Z}[x,s,t]/(st-1) = \mathbb{Z}[x,t,t^{-1}]$, as well as to $\mathbb{Z}[y,s,t]/(st-1) = \mathbb{Z}[y,s,s^{-1}]$. Since this ring is a localization of both $\mathbb{Z}[x,t]$ and $\mathbb{Z}[y,s]$, we can identity its spectrum both as an open subset of $\operatorname{Spec}\mathbb{Z}[x,t]$ and as an open subset of$\operatorname{Spec}\mathbb{Z}[y,s]$. This gives a diagram

$$\begin{tikzcd}
 & \mathbb{A}^2 \arrow[d, equal] & \\
 & \operatorname{Spec} \mathbb{Z}[x,y] & \\
U = \operatorname{Spec} \mathbb{Z}[x,t] \quad \mathbb{A}^2 \arrow[ru] & & \operatorname{Spec} \mathbb{Z}[y,s] = V \quad \mathbb{A}^2 \arrow[lu] \\
 & \operatorname{Spec} R \arrow[lu, hook] \arrow[ru, hook] &
\end{tikzcd}$$

where the bottom diagonal maps are the two open immersions. Hence we

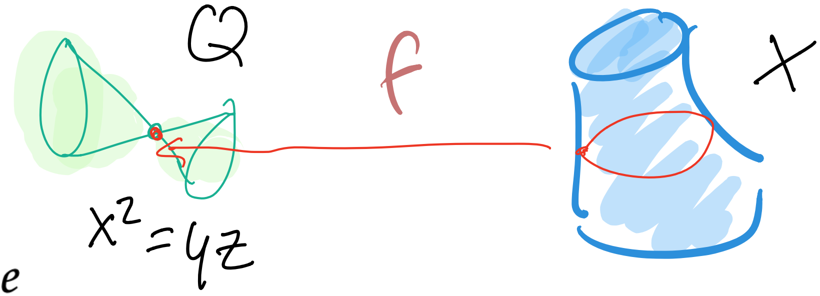
$$X = \mathbb{A}^2 \cup_{\operatorname{Spec} R} \mathbb{A}^2$$

where the bottom diagonal maps are the two open immersions. Hence we can glue these two affine spaces together along $\operatorname{Spec} R$ to obtain a new scheme X. By construction, the restriction of the maps $\operatorname{Spec} \mathbb{Z}[x,t] \to \operatorname{Spec} \mathbb{Z}[x,y]$ and $\operatorname{Spec} \mathbb{Z}[y,s] \to \operatorname{Spec} \mathbb{Z}[x,y]$ to $\operatorname{Spec} R$ coincide with the map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}[x,y]$ which is induced by $\mathbb{Z}[x,y] \to R$. Therefore they glue together to a morphism

$$p : \mathrm{X} \to \mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x,y].$$

$$\begin{tikzcd}
X \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P}^1 \\
\mathbb{A}^2 &
\end{tikzcd}$$

To complete the discussion, we should define the corresponding morphism $q : X \to \mathbb{P}^1$. Again we work locally. On the affine open $U = \operatorname{Spec} \mathbb{Z}[x, t]$, we have a map $U \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[t]$ induced by the inclusion $\mathbb{Z}[t] \subset \mathbb{Z}[x, t]$. Similarly, on $V = \operatorname{Spec} \mathbb{Z}[y, s]$, we have a map $V \to \mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[s]$. To check if they can be glued together, we have to see what happens on the overlap $U \cap V = \operatorname{Spec} R$. However, on $\operatorname{Spec} R$ it holds that $t = s^{-1}$, so using the standard description of $\mathbb{P}^1$ as glued together of two affine lines, we see that the maps $\mathbb{Z}[t] \to R$ and $\mathbb{Z}[s] \to R$ induce the desired morphism $q : X \to \mathbb{P}^1$.



Resolution of a quadric cone

Let k be a field, and consider the following diagram of inclusions of subrings of $k[x^{\pm 1}, y^{\pm 1}]$.

$$\begin{tikzcd}
 & k[x,y,x^2y^{-1}] \arrow[dl, hook] \arrow[dr, hook] & \\
\underset{\mathbb{A}^2}{k[x^{-1}y, x^2y^{-1}]} \arrow[dr, hook] & & \underset{\mathbb{A}^2}{k[xy^{-1}, y]} \arrow[dl, hook] \\
 & k[xy^{-1}, x^{-1}y, y] &
\end{tikzcd}$$

$Q \cong \dfrac{k[x,y,z]}{(x^2-yz)}$

$z \mapsto x^2y^{-1}$

Note that there is an isomorphism of k-algebras $k[x,y,z]/(x^2-yz) \to k[x,y,x^2y^{-1}]$, sending z to x^2y^{-1}. Thus applying Spec, we obtain a diagram of schemes

$$\begin{tikzcd}
& Q = \operatorname{Spec} k[x,y,z]/(x^2-yz) & \\
\mathbb{A}^2_k \arrow[ur] & & \mathbb{A}^2_k \arrow[ul] \\
& U = \operatorname{Spec} k[xy^{-1}, x^{-1}y, y] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\rightsquigarrow \quad X = \mathbb{A}^2 \cup_{U} \mathbb{A}^2$$

Note that $k[xy^{-1}, x^{-1}y, y]$ is a localization of both $k[x^{-1}y, x^2y^{-1}]$ and $k[xy^{-1}, y]$ (we invert $x^{-1}y$ and xy^{-1} respectively). Thus U lies naturally as a distinguished open set in both $\mathbb{A}^2_k$'s. The lower part of the diagram then allows us to glue the two copies of $\mathbb{A}^2_k$ over the open set U to a new scheme X. The top part of the diagram shows that the two morphisms $f_1 : \mathbb{A}^2_k \to Q$ and $f_2 : \mathbb{A}^2_k \to Q$ glue to a morphism $f : X \to Q$.

5.2 *Projective space*

We now give examples of more involved gluings. Let A be a ring, and consider the subrings of $A[x_0^{\pm 1}, \dots x_n^{\pm 1}]$ given by

$$R_i = A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$$

$n+1$

$\hookrightarrow \operatorname{Spec} R_i \simeq \mathbb{A}^n_A$

for $i = 0, \dots, n$. Note that we have equalities

$$R_i\left[\left(\frac{x_j}{x_i}\right)^{-1}\right] = R_j\left[\left(\frac{x_i}{x_j}\right)^{-1}\right]$$

$$R_i\left[\left(\frac{x_j}{x_i}\right)^{-1}\right] = R_j\left[\left(\frac{x_i}{x_j}\right)^{-1}\right]$$

for each i and j. Thus we can glue the affine spaces $X_i = \operatorname{Spec} R_i \simeq \mathbb{A}_i^n$ together to a scheme which we will denote by $\mathbb{P}_A^n$. This is the *projective n-space over A*.

Note that each Spec R_i come with a canonical map Spec $R_i \to$ Spec A, induced by the inclusion $A \subset R_i$. Moreover, the isomorphisms above are all 'over A', thus compatible with these inclusions, and we see that we may glue to form a morphism $\mathbb{P}^n_A \to \operatorname{Spec} A$.

Note in particular, that for $n = 1$ we obtain the $\mathbb{P}^1_A$ constructed earlier. An argument similar to that in Proposition 5.1 gives

Proposition 5.2 $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$

alg.geo.1 :

$U_0 = D(x_0)$ $\quad x = \frac{x_1}{x_0}, \frac{x_2}{x_0} = y$

$U_1 = D(x_1)$ $\quad \frac{x_0}{x_1} = x^{-1} \quad \frac{x_2}{x_1} = yx^{-1}$

Example 5.3 The *projective plane* $\mathbb{P}^2_k$ is the scheme glued together by the 3 affine planes $\mathbb{A}^2_k$:

$$U_0 = \operatorname{Spec} k[x, y], \quad U_1 = \operatorname{Spec} k[x^{-1}, yx^{-1}], \quad U_2 = \operatorname{Spec} k[y^{-1}, xy^{-1}].$$

$Y \hookrightarrow X$ underskjema $\Longleftrightarrow$ $(Y, \mathcal{O}_Y) \simeq \left(Y, \frac{\mathcal{O}_X}{\mathcal{J}}\right)$

$\mathcal{J} \subset \mathcal{O}_X$

Consider the 3 ideals

$$I_0 = (y^2 - x^3) \subset k[x, y]$$
$$I_1 = (x^{-1})(yx^{-1})^2 - 1) \subset k[x^{-1}, yx^{-1}]$$
$$I_2 = (y^{-1} - (xy^{-1})^3) \subset k[y^{-1}, xy^{-1}].$$

Each ideal I_i defines a closed subscheme of the corresponding $U_i = \mathbb{A}^2_k$, and it is readily checked that they agree on the overlaps $U_i \cap U_j$. For instance, in $U_0 \cap U_1 = \operatorname{Spec} k[x^{\pm 1}, y]$, we have

$$\Big((x^{-1})(yx^{-1})^2 - 1\Big) = (x^{-3}(y^2 - x^3)) = (y^2 - x^3),$$

since x is invertible. Thus the three glue to a closed subscheme $Z \subset \mathbb{P}^2_k$.

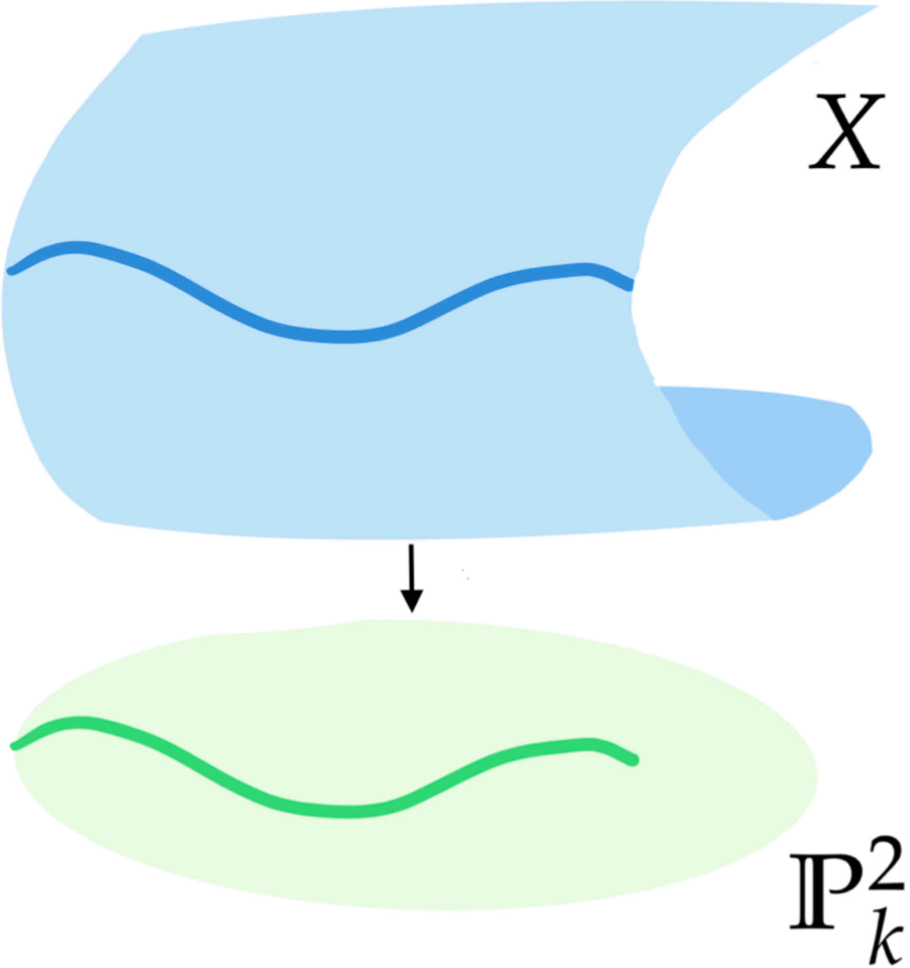
In Chapter 7, we will see that there is a much more economic way of specifying subschemes of $\mathbb{P}^n$, using graded ideals. In fact, the above subscheme is defined by a single homogeneous polynomial, $F = x_0 x_2^2 - x_1^3$.

Double covers of $\mathbb{P}^n_A$

We may similarly generalize the example of hyperelliptic curves to higher dimensions: Let A be a ring and let $R = A[x_0, \ldots, x_n]$. Let $f \in R$ be a homogeneous polynomial of degree $2d$, and let

$$S_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{y}{x_i^d}\right] \Big/ \left(\left(\frac{y}{x_i^d}\right)^2 - f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right)$$

It is straightforward to verify that $\operatorname{Spec} S_i$ glue to a scheme X. Moreover, keeping the notation R_i from the previous section, the morphisms $\operatorname{Spec} S_i \to \operatorname{Spec} R_{(x_i)}$ glue to a morphism $\pi : X \to \mathbb{P}^n_A$.


X
$\mathbb{P}^2_k$

Example 5.4 *(A Del Pezzo surface)* Let us consider the case $f(x_0, x_1, x_2) = x_1^4 + x_0^3 x_1 + x_2^2(x_2 - x_0)^2$. Note that

$$S_0 \simeq k[u, v, y]/(y^2 - u^3 - u + v^2(v^2 - 1))$$

via the identifications $u = \frac{x_1}{x_0}$, $v = \frac{x_2}{x_0}$. So the scheme X is a surface glued out of three open sets, each isomorphic to a quartic surface in $\mathbb{A}_k^3$. The ‘double cover’ morphism is given by $\pi : \operatorname{Spec} S_0 \to \operatorname{Spec} k[u, v]$.

The closed subset $V(u)$ is interesting: Note that

$$(y^2 - u^4 - u + v^2(v-1)^2, u) = (y + v(v-1), u) \cap (y - v(v-1), u)$$

So the preimage $\pi^{-1}(V(u))$ consists of two components, each mapping isomorphically to $V(u)$. ☆

Hirzebruch surfaces

Let $r \geqslant 0$ be an integer and consider the scheme X which is glued together by the four affine scheme charts

$$\begin{aligned} U_{00} &= \operatorname{Spec} k[x, y] & U_{01} &= \operatorname{Spec} k[x, y^{-1}] \\ U_{10} &= \operatorname{Spec} k[x^{-1}, x^r y] & U_{11} &= \operatorname{Spec} k[x^{-1}, x^{-r} y^{-1}] \end{aligned} \tag{5.2}$$

The inclusions

$$\begin{array}{cc} k[x] \subset k[x,y] & k[x] \subset k[x,y^{-1}] \\ k[x^{-1}] \subset k[x^{-1}, x^r y] & k[x^{-1}] \subset k[x^{-1}, x^{-r} y^{-1}] \end{array} \tag{5.3}$$

induce morphisms $U_{ij} \to \mathbb{A}^1_k$. Moreover, these agree over the various intersections $U_{ij} \cap U_{jl}$, and so we obtain a morphism $X \to \mathbb{P}^1_k$.