Chapter 6

Geometric properties of schemes

6.1 *Decomposition into irreducible subsets*

6.1 Let A be a ring and consider a primary decomposition of the ideal $\mathfrak{a}$:

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r.$$

Putting $Y_i = V(\sqrt{\mathfrak{q}_i})$, we find $V(\mathfrak{a}) = Y_1 \cup Y_2 \cup \cdots \cup Y_r$, where each Y_i is an irreducible closed set in $\mathbb{A}^n$. If the prime $\sqrt{\mathfrak{q}_i}$ is not minimal among the associated primes, say $\sqrt{\mathfrak{q}_j} \subset \sqrt{\mathfrak{q}_i}$, it holds that $Y_i \subset Y_j$, and the component Y_i contributes nothing to intersection and can be discarded.

6.2 In a more general context, a decomposition $Y = Y_1 \cup \cdots \cup Y_r$ of any topological space is said to be *redundant* if one can discard one or more of the Y_i's without changing the union. That a component Y_j can be omitted is equivalent to Y_j being contained in the union of rest; that is, $Y_j \subset \bigcup_{i \neq j} Y_i$. A decomposition that is not redundant, is said to be *irredudant*. Translating the Noether–Lasker theorem into geometry we arrive at the following:

Proposition 6.3 *If A is a noetherian ring, any closed subset $Y \subset \operatorname{Spec} A$ can be written as an irredundant union*

$$Y = Y_1 \cup \cdots \cup Y_r$$

where the Y_i's are irreducible closed algebraic subsets. The union is unique up to the order of the Y_i's.

Notice that since embedded components do not show up for radical ideals, we get a clear and clean uniqueness statement.

Example 6.5 Consider the closed set $Y = V(I) \subset \mathbb{A}^3$ given by the ideal

$$I = (x^2 - y, xz - y^2, x^3 - xz)$$

Note first that if $x = 0$, then $y = 0$, so $V(x,y) \subset X$. If $x \neq 0$, the third equation gives $z = x^2$, and so by the first and second equations we get $xz - y^2 = x^3 - x^4$, giving $x = 1$, $y = 1$ and $z = 1$. Hence

$$X = V(x,y) \cap V(x-1, y-1, z-1)$$

That is, X is the union of the z-axis, and the point $(1,1,1)$. In fact, a primary decomposition of I is given by $I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3$, where

$$\mathfrak{q}_1 = (x, y), \quad \mathfrak{q}_2 = (x-1, y-1, z-1), \quad \mathfrak{q}_3 = (x^2 - y, xy, y^2, z).$$

Taking radicals, we find that the primes associated to I are

$$\mathfrak{p}_1 = (x, y), \quad \mathfrak{p}_2 = (x-1, y-1, z-1), \quad \mathfrak{p}_3 = (x, y, z).$$

Note that $\mathfrak{p}_1 \subset \mathfrak{p}_3$, so $\mathfrak{p}_3$ is an embedded component, so it does not show up in the decomposition above. ☆

6.7 A decomposition result as in Proposition 6.3 above holds for a much broader class of topological spaces than the closed sets. The class in question is the class of the so-called *Noetherian topological spaces*; these comply to the requirement that every descending chain of closed subsets is eventually stable. That is; if $\{X_i\}$ is a collection of closed subsets forming a chain

$$\ldots X_{i+1} \subset X_i \subset \ldots \subset X_2 \subset X_1,$$

it holds true that for some index r one has $X_i = X_r$ for $i \geqslant r$.

6.8 The Noether–Lasker decomposition of closed subsets in affine space as a union of irreducibles can be generalized to any Noetherian topological space:

Theorem 6.9 *Every closed subset Y of a Noetherian topological space X has an irredundant decomposition $Y = Y_1 \cup \cdots \cup Y_r$ where each is Y_i is a closed and irreducible subset of X. Furthermore, the decomposition is unique up to order.*

6.2 *Noetherian schemes*

By the correspondence between irreducible subsets of Spec A and prime ideals of A, we immediately see that if A is a Noetherian ring, the prime spectrum Spec A is a Noetherian topological space.

$$Y_1 \supset Y_2 \supset Y_3 \supset \cdots$$

$$V(a_1) \supset V(a_2) \supset V(a_3) \supset \cdots \quad \rightsquigarrow \quad a_1 \subset a_2 \subset \cdots$$

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The converse fails:

$$A = \frac{k[t_1, t_2, \ldots]}{(t_1 t_2, t_1 t_3, \ldots)}$$

Example 6.10 Consider the polynomial ring $k[t_1, t_2, \ldots]$ in countably many variables t_i and mod out by the square $\mathfrak{m}^2$ of the maximal ideal generated by the variables, $\mathfrak{m} = (t_1, t_2, \ldots)$. The resulting ring A has just one prime ideal, the one generated by the t_i's. So Spec A has just one point, and hence is noetherian. The ring A, however, is clearly not Noetherian; the sole prime ideal requires infinitely many generators, namely all the t_i's. ☆

$$\operatorname{Spec}(A/I) \cong V(I) \subseteq \operatorname{Spec} A$$

$$V(\mathfrak{m}) = V(\mathfrak{m}^2) = \{(\mathfrak{m})\}$$

In light of this example, we take a different route to define noetherianness for schemes:

Definition 6.11 *i) A scheme is* locally Noetherian *if it can be covered by open affine subsets* $\operatorname{Spec} A_i$ *where each* A_i *is a Noetherian ring*

ii) A scheme is Noetherian *if it is both locally Noetherian and quasi-compact.*

Recall from Chapter 3 that a scheme X is *quasi-compact* if every open cover of X has a finite subcover. We also showed that affine schemes were quasi-compact: Any open covering can be refined to a covering by distinguished open sets $D(f_i)$, and when $\operatorname{Spec} A = \bigcup_i D(f_i)$, the ideal generated by the f_i's contains 1, and the finitely many $D(f_i)$'s with f_i occurring in an expansion of 1, will do.

From the definition, it follows that a general scheme is Noetherian if and only if it can be covered by finitely many open affines Spec A_i where each A_i is Noetherian.

In fact, with the new definition, we now have

Proposition 6.12 Spec A *is Noetherian (as a scheme) if and only if* A *is Noetherian.*

You should think of this as a purely algebraic fact: Refining the cover, we may assume that each $A_i = A_{f_i}$. By a theorem in commutative algebra, a ring A is Noetherian provided that each localization A_{f_i} is Noetherian and $1 \in (f_1, \ldots, f_r)$.

Example 6.13 Let k be a field. The following schemes are not Noetherian:

i) $\coprod_{i=1}^{\infty} \mathbb{A}_k^1$;

ii) $\operatorname{Spec} \bigoplus_{i=1}^{\infty} k[x]$;

iii) $\operatorname{Spec} \prod_{i=1}^{\infty} k[x]$.

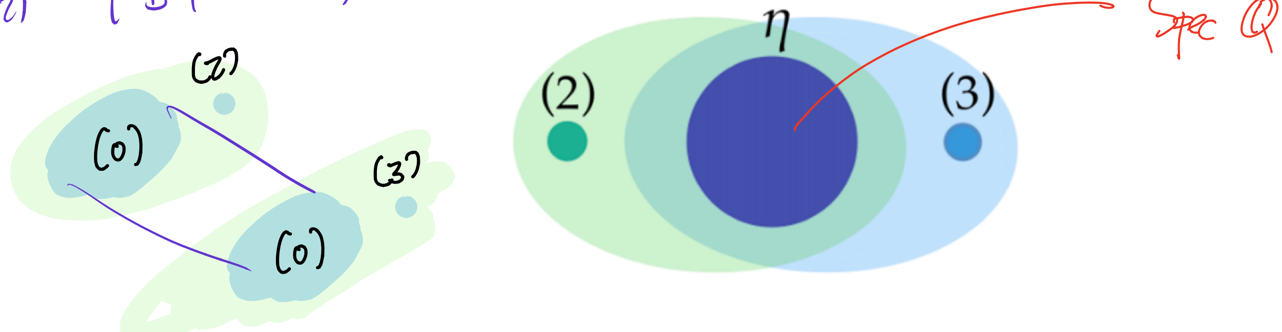
where the union is the disjoint union. We also remark that these are different: the disjoint union $\coprod_{i=1}^{\infty} \mathbb{A}_k^1$ is not quasi-compact (thus not affine). The latter two are affine (thus quasi-compact), but non-isomorphic, since their rings of global sections are non-isomorphic.

Proposition 6.14 *If X is a Noetherian scheme, then its underlying topological space is Noetherian.*

Proof: Since X is quasi-compact it may be covered by a finite number of open affine subsets, and since a descending chain stabilizes if the intersection with each of those open sets stabilizes, it suffices to show the proposition for $X = \operatorname{Spec} A$ with A a Noetherian ring. In that case a descending chain of closed subsets is of the form $V(\mathfrak{a}_1) \supset V(\mathfrak{a}_2) \supset \cdots$, where we may assume that the ideals $\mathfrak{a}_n$ are radical. Then the condition that $V(\mathfrak{a}_n)$ is decreasing, corresponds to the sequence $(\mathfrak{a}_n)$ being increasing, and so it has to be stationary because A is Noetherian . ❏

Proposition 6.15 *Let X be a (locally) Noetherian scheme. Then any closed or open subscheme of X is also (locally) Noetherian.*

$\mathbb{Z}_{(2)} = \{ \frac{a}{b} \mid 2 \nmid b \} \subset \mathbb{Q}$



Example 5.2 *(Semi-local rings)* The rings $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$ are both discrete valuation rings whose maximal ideals are (2) and (3) respectively. Their fraction fields are both equal to $\mathbb{Q}$. Let $X_1 = \operatorname{Spec} \mathbb{Z}_{(2)}$ and $X_2 = \operatorname{Spec} \mathbb{Z}_{(3)}$. Both have a generic point that is open, so there is a canonical open immersion $\operatorname{Spec} \mathbb{Q} \to X_i$ for $i = 1, 2$. Hence we can glue the two along their generic points and thus obtain a scheme X with one open point η and two closed points.

$X_1 = \operatorname{Spec} \mathbb{Z}_{(2)}$

$X_2 = \operatorname{Spec} \mathbb{Z}_{(3)}$

Let us compute the global sections of $\mathcal{O}_X$ using the now classical sequence for the open covering $\{X_1, X_2\}$:

$$\begin{array}{ccccc}
\Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(X_1, \mathcal{O}_X) \times \Gamma(X_2, \mathcal{O}_X) & \longrightarrow & \Gamma(X_1 \cap X_2, \mathcal{O}_X) \\
 & & {\scriptstyle =}\big\downarrow & & \big\downarrow \\
 & & \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} & \xrightarrow[\rho]{\hspace{2em}} & \mathbb{Q}.
\end{array}$$

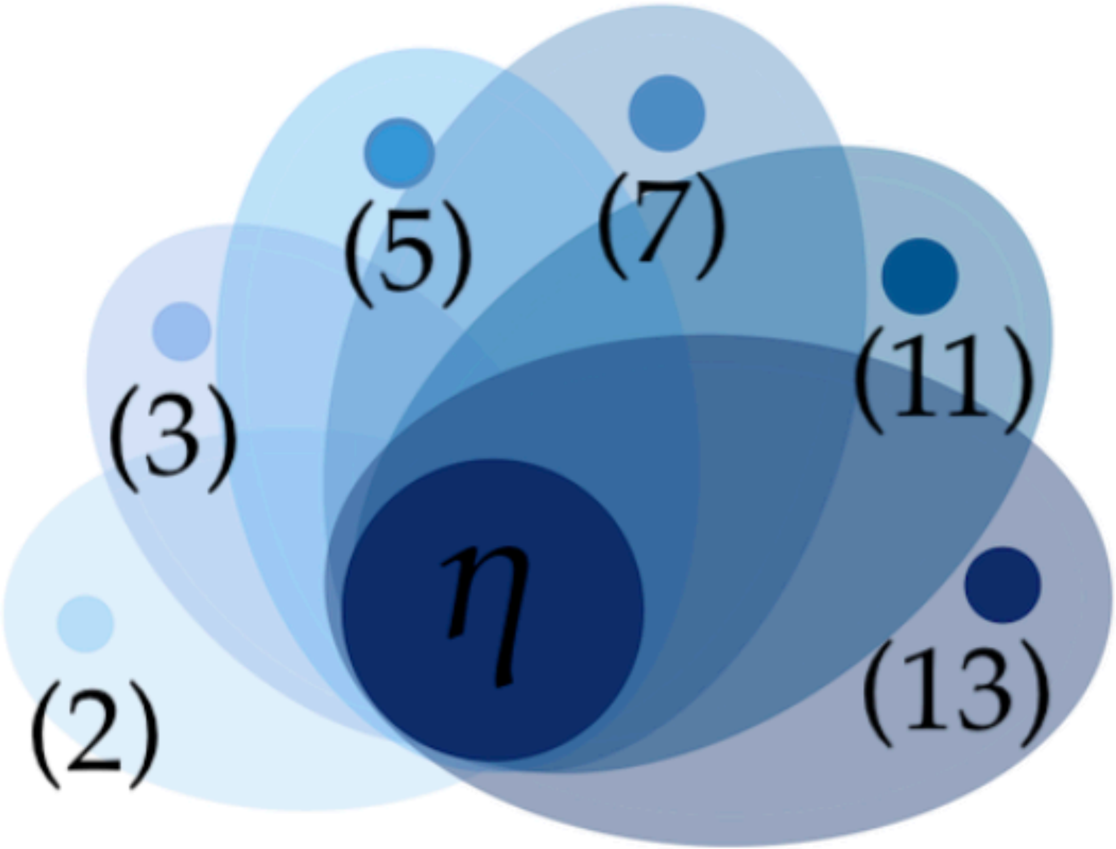
The map ρ sends a pair (an^{-1}, bm^{-1}) to the difference $an^{-1} - bm^{-1}$, hence the kernel consists of the diagonal, so to speak, in $\mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)}$, which is isomorphic to the intersection $\mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$. This is a semi-local ring with the two maximal ideals (2) and (3). Hence there is a map $X \to \operatorname{Spec} \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ and it is left as an exercise to show this is an isomorphism.

EXAMPLE 5.3 *(More semi-local rings)* More generally, if $P = \{p_1, \ldots, p_r\}$ is a finite set of distinct prime numbers, one may let $X_p = \operatorname{Spec} \mathbb{Z}_{(p)}$ for $p \in P$. There is, as in the previous case, canonical open embedding $\operatorname{Spec} \mathbb{Q} \to X_p$. Let the image be $\{\eta_p\}$. Obviously the gluing conditions are all satisfied (the transition maps are all equal to $\mathrm{id}_{\operatorname{Spec} \mathbb{Q}}$ and $X_{pq} = \{\eta_p\}$ for all p). We do the gluing and obtain a scheme X.

$\frac{a}{b}$ b the deleting mod non p_i

$\Gamma(X, \mathcal{O}_X) = \bigcap_{p \in P} \mathbb{Z}_{(p_i)}$ semilocal ring

$X \simeq \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$


(2)
(3)
(5)
(7)
(11)
(13)
η

Example 6.16 In Example 5.3, we worked with a finite set of primes, but the hypothesises of the gluing theorem impose no restrictions on the number of schemes to be glued together, and we are free to take $\mathcal{P}$ infinite, for example we can use the set $\mathcal{P}$ of all primes! The glued scheme $X_{\mathcal{P}}$ is a peculiar animal: it is neither affine nor Noetherian, but it is locally Noetherian.

In this case, we have $\bigcap_{p \in \mathcal{P}} \mathbb{Z}_{(p)} = \mathbb{Z}$, so

$\Gamma(X, \mathcal{O}_X) = \mathbb{Z}$

There is a map $\phi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ which is bijective and continuous, but not a homeomorphism, and it has the property that for all open subsets $U \subset \operatorname{Spec}\mathbb{Z}$ the map induced on sections $\phi^\sharp\colon \Gamma(U, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}) \to \Gamma(\phi^{-1}U, \mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism, in other words, $\phi^\sharp\colon \mathcal{O}_{\operatorname{Spec}\mathbb{Z}} \to \phi_*(\mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism!

As before we construct the scheme $X_{\mathcal{P}}$ by gluing the different $\operatorname{Spec}\mathbb{Z}_{(p)}$'s together along the generic points. However, when computing the global sections, we see things changing. The kernel of ρ is still $\bigcap_{p\in\mathcal{P}}\mathbb{Z}_{(p)}$, but now this intersection equals $\mathbb{Z}$: indeed, a rational number $\alpha = a/b$ lies in $\mathbb{Z}_{(p)}$ precisely when the denominator b does not have p as factor, so lying in all $\mathbb{Z}_{(p)}$, means that b has no non-trivial prime-factor. That is, $b = \pm 1$, and hence $\alpha \in \mathbb{Z}$.

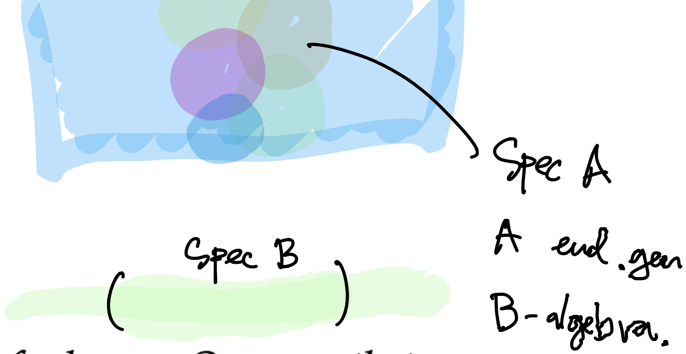
There is a morphism $X_{\mathcal{P}} \to \operatorname{Spec} \mathbb{Z}$ which one may think about as follows. Each of the schemes $\operatorname{Spec} \mathbb{Z}_{(p)}$ maps in a natural way into $\operatorname{Spec} \mathbb{Z}$, the mapping being induced by the inclusions $\mathbb{Z} \subset \mathbb{Z}_{(p)}$. The generic points of the $\operatorname{Spec} \mathbb{Z}_p$'s are all being mapped to the generic point of $\operatorname{Spec} \mathbb{Z}$. Hence they patch together to give a map $X_{\mathcal{P}} \to \operatorname{Spec} \mathbb{Z}$. This is a continuous bijection by construction, but it is not a homeomorphism: indeed, the subsets $\operatorname{Spec} \mathbb{Z}_{(p)}$ are open in $X_{\mathcal{P}}$ by the gluing construction, but they are not open in $\operatorname{Spec} \mathbb{Z}$, since their complements are infinite, and the closed sets in $\operatorname{Spec} \mathbb{Z}$ are just the finite sets of maximal ideals.

The topology of the scheme $X_{\mathcal{P}}$ is not Noetherian since the subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ form an open cover that obviously can not be reduced to a finite cover. However, it is locally Noetherian, as the open subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ are Noetherian. The sets $U_p = X_{\mathcal{P}} \setminus \{(p)\}$ map bijectively to $D(p) \subset \operatorname{Spec}\mathbb{Z}$ and $\Gamma(U_p, \mathcal{O}_{X_{\mathcal{P}}}) = \mathbb{Z}_p$, but U_p and $D(p)$ are not isomorphic. ☆

6.3 *Other finiteness properties*

Noetherian rings mostly behave well, but they can be elusive and there are specimens among them that show a weird behaviour. There are stronger finiteness conditions that makes schemes have many of the agreeable properties of varieties.

Recall that giving a morphism $f\colon X \to S$ between two affine schemes $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$, is equivalent to giving the ring homomorphism $f^\sharp\colon A \to B$, or said differently giving B the structure of an A-algebra.



DEFINITION 6.17 *Let $f\colon X \to Y$ be a morphism of schemes. One says that:*

i) *f is of* locally finite type *if Y has a cover consisting of open affine subsets $V_i = \operatorname{Spec} B_i$ such that each $f^{-1}(V_i)$ can be covered by affine subsets of the form $\operatorname{Spec} A_{ij}$, where each A_{ij} is finitely generated as a B_i-algebra.*

ii) *f is of* finite type *if, in i), one can do with a finite number of $\operatorname{Spec} A_{ij}$.*

In case $S = \operatorname{Spec} A$, one says that a scheme over A is of locally finite type *(respectively of* finite type*) over A, if the morphism $X \to \operatorname{Spec} A$ is locally of finite type (respectively of finite type).*

ex $\quad f: \mathbb{A}^1_k \longrightarrow \operatorname{Spec} k \qquad k[x]$ fin. k-alg.

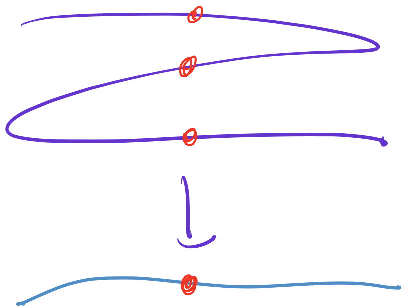
ex $\quad f: \mathbb{P}^1_k \longrightarrow \operatorname{Spec} k \qquad$ } fin. type

Again, when $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, the scheme X is of finite type over A precisely when $B = A[x_1, \ldots, x_n]/\mathfrak{a}$ for an ideal $\mathfrak{a}$. One easily checks that both closed and open immersions are of finite type.

There is another related, but much stronger finiteness property a morphism can have:

Definition 6.18 *A morphism $f\colon X \to Y$ is* finite *if there is a covering $V_i = \operatorname{Spec} A_i$ such that each inverse image $f^{-1}(V_i)$ is affine, an if $f^{-1}(V_i) = \operatorname{Spec} B_i$, the A_i-algebra B_i is a finite A_i-module.*

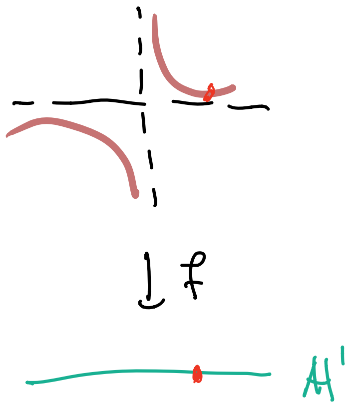
To underline the huge difference between the the two notions, note that a scheme X which is finite over a field k, in particular has a finite and discrete underlying topological space, whereas X being of finite type, merely means it is covered by affine schemes of the form $\operatorname{Spec} k[x_1, \ldots, x_r]/\mathfrak{a}$. This generalizes in the following way:



Proposition 6.19 *A finite morphism has scheme-theoretical finite fibres. In particular, the fibres are finite discrete topological spaces.*

Proof: We may certainly assume that both X and Y are affine; say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Any generator set of B as an A-module, persists being a generator set of $B \otimes_A K(A/\mathfrak{p})$ as a vector space over $K(A/\mathfrak{p})$, where $\mathfrak{p} \in \operatorname{Spec} A$ is any point. ❑

Be aware that the converse is far from being true. One easily finds so-called *quasi-finite* morphisms; that is, morphisms with all fibres finite, that are not



$\operatorname{Spec} \dfrac{k[x,y]}{(xy-1)}$

$\operatorname{Spec} k[x]$

f ikke endelig

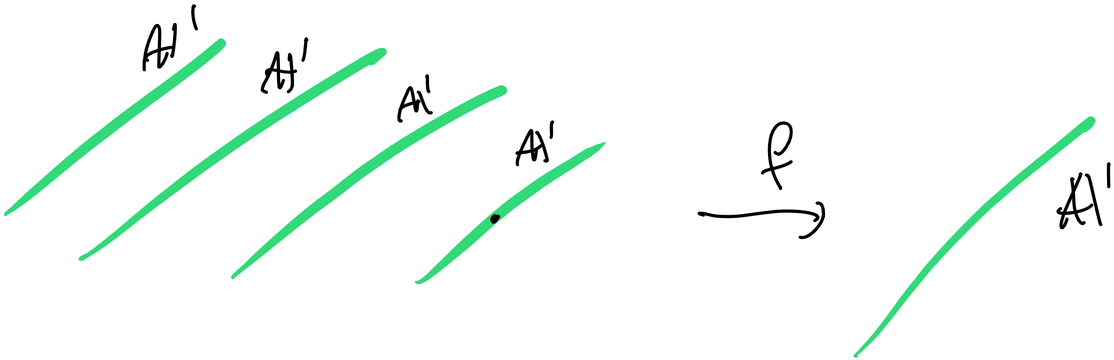
$k[x,y]/(xy-1) \simeq k[x,x^{-1}]$ ikke endelig $k[x]$-modul.

finite: every injective morphism is evidently quasi-finite, so for instance open immersions will be, and open immersions are not finite except in trivial cases. The arch-type is the inclusion $\iota\colon D(x) \hookrightarrow \mathbb{A}^1_k$ which on the ring level corresponds to the inclusion $k[x] \hookrightarrow k[x, x^{-1}]$; and $k[x, x^{-1}]$ is not a finite module over $k[x]$. We'll come back to the relation between quasi-finite and finite morphism when having introduced proper morphism (in Section 16.2).

Examples

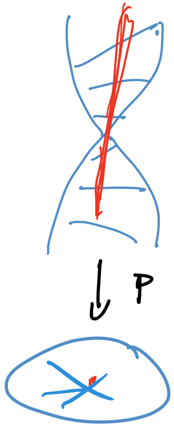
6.20 For $n \geqslant 1$, the structure morphisms $\mathbb{A}^n_k \to \operatorname{Spec} k$ and $\mathbb{P}^1_k \to \operatorname{Spec} k$ are of finite type, but not finite.

$k[X_1 \ldots X_n]$ er ikke en endelig k-modul.

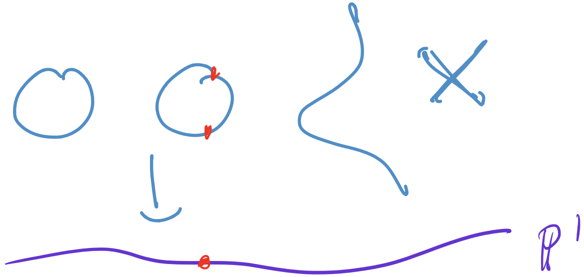


6.21 The morphism $\coprod_{i=1}^{\infty} \mathbb{A}_k^1 \to \mathbb{A}_k^1$ (identity on each component) is locally of finite type, but not of finite type.

$$\mathbb{Z}[x,y] \longrightarrow \mathbb{Z}[x,t], \quad x \mapsto x, \quad y \mapsto xt$$



6.22 Consider the blow-up morphism $\pi : X \to \mathbb{A}^2$ from Example 5.2. In the local charts, π is given by $\operatorname{Spec}\mathbb{Z}[x,t] \to \operatorname{Spec}\mathbb{Z}[x,y]$ induced by $y \mapsto xt$, making $\mathbb{Z}[x,t]$ into a finitely generated $\mathbb{Z}[x,t]$-algebra. However, it is not finite, since $\pi^{-1}(V)$ contains a copy of $\mathbb{P}^1$ for any neighbourhood V of the closed point $o \in \mathbb{A}^2$, which is not possible for affine schemes.



6.23 Let us revisit the example of a hyperelliptic curve X from Section 5.1. In the notation from that section, the curve X has an open covering consisting of two affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ and there is a 'double cover' morphism $f : X \to \mathbb{P}^1_k$. This is a finite morphism: Over U it is induced by the inclusion

$$k[x] \subset \frac{k[x,y]}{(y^2 - a_{2g+1}x^{2g+1} - \cdots - a_1 x)},$$

and the algebra on the right is isomorphic to $k[x] \oplus k[x]y$ as a $k[x]$-module. A similar statement holds for the morphism $f|_V : V \to \mathbb{A}^1_k$, so f is a finite morphism. ☆

6.4 *The dimension of a scheme*

Recall that the *Krull dimension* of a ring A is defined as the suprenum of the length of all chains of prime ideals in A. For a scheme, we make the following similar definition:

Definition 6.24 *Let X be a scheme. The dimension of X is the suprenum of all integers n such that there exists a chain*

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct closed irreducible closed subsets of X.

Note that this suprenum might not be a finite number, in which case we say that $\dim X = \infty$. Note also that the dimension of X only depends on the underlying topological space. In particular, $\dim X = \dim X_{\text{red}}$.

In the case where $X = \operatorname{Spec} A$ is affine, we know that the closed irreducible subsets are of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal. Using this observation we find

Proposition 6.25 *The dimension of $X = \operatorname{Spec} A$ equals the Krull dimension of A.*

Example 6.26

i) The dimension of $\mathbb{A}^n_A = \operatorname{Spec} A[x_1, \ldots, x_n]$ is $n + \dim A$ when A is a Noetherian ring (for general rings $\dim \mathbb{A}^n_A$ is comprised between $\dim A + n$ and $\dim A + 2n$, and all values are possible) In particular, when $A = k$ is a field, $\mathbb{A}^n_k$ has dimension n. A maximal chain is $V(x_1) \supset V(x_1, x_2) \supset \cdots \supset V(x_1, \ldots, x_n)$.

ii) $\dim \operatorname{Spec} \mathbb{Z}$ is 1. All maximal chains have the form $V(p) \subset V(0) = \operatorname{Spec} \mathbb{Z}$.

iii) $\dim \operatorname{Spec}(k[\epsilon]/\epsilon^2) = \dim \operatorname{Spec} k = 0$.

Remark 6.27 *Having finite dimension does not guarantee that the scheme is Noetherian. The quotient of $\mathbb{Q}[x_1, x_2, \ldots]$ by the ideal generated by all products $x_i x_j$ with $1 \leqslant i \leqslant j < \infty$ is an example. Here there is only one prime ideal (generated by all the variables), but the scheme is clearly not Noetherian.*

There are even Noetherian rings whose Krull dimension is infinite.

Definition 6.28 *Let $Y \subset X$ be a closed subset of X. We define the* codimension *of Y as the supremum of all integers n such that there exists a chain*

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of X.

The codimension of $V(\mathfrak{p})$ in $\operatorname{Spec} A$ is the height of the prime $\mathfrak{p}$ in A.

One should have in mind that that codimension can be contra-intuitive even for Noetherian schemes; for instance, there are Noetherian affine schemes of any dimension with closed points being of codimension one; we shall see a two-dimensional one in Proposition 23.22.

For integral schemes of finite type over fields, we can study the dimension in terms of the fraction field:

Theorem 6.29 *Let X be an integral scheme of finite type over a field k, with function field K. Then*

i) The dimension $\dim X$ equals the trancendence degree of K over k (in particular, $\dim X < \infty$);

ii) For each $U \subset X$ open, $\dim U = \dim X$;

iii) If $Y \subset X$ is a closed subset, then $\operatorname{codim} Y = \inf\{\dim \mathcal{O}_{X,p} | p \in Y\}$ and

$$\dim Y + \operatorname{codim} Y = \dim X.$$

In particular, for a closed point $p \in X$, $\dim X = \dim \mathcal{O}_{X,p}$.

Examples

6.30 The scheme $\mathbb{P}^n_k$ satisfies the conditions of the theorem. It's dimension is n, which follows because $\mathbb{P}^n_k$ contains $\mathbb{A}^n_k$ as an open dense subset, and $\mathbb{A}^n_k$ has dimension n.

6.31 The quadric cone $Q = \operatorname{Spec} k[x, y, z]/(x^2 - yz)$ of Section 5.2 has dimension 2. This follows as the function field $K(Q) = k(y, z)$ has transcendence degree 2 over k. Alternatively, we can use the morphisms $f_i : \mathbb{A}^2_k \to Q$ which are isomorphisms over an open set $U \subset \mathbb{A}^2_k$ (which thus also has dimension 2).

6.32 Its important to note that the formula $\dim Y + \operatorname{codim} Y = \dim X$ does not always hold, even if X is the spectrum of a very nice ring. Indeed, let $X = \operatorname{Spec} R[t]$ where R is any DVR with generator t of the maximal ideal, for instance, the localization $R = k[t]_{(t)}$. The prime $\mathfrak{p} = (tu - 1)$ has height one, but $A/\mathfrak{p} \simeq R[1/t]$ is a field, hence of dimension zero. However, $\dim A = \dim R + 1 = 2$. ☆

For schemes which aren't integral but still of finite type, we still have a good control over its dimension. First of all, the dimension of X is the same as of X_{red}, so we may assume that X is reduced. Then, if $X = \bigcup X_i$ is the decomposition into irreducible components, we have that X_i is integral, and $\dim X$ is the supremum of all $\dim X_i$.

Example 6.33 Consider $X = \mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ and $Y = V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = \left(xy - x, x^2, y^2z - z, y^3 - y, xy^2 - xy\right) = (z,y,x) \cap \left(y-1, x^2\right) \cap (y+1, x)$$

The associated primes of $\mathfrak{a}$ are $\mathfrak{p}_1 = (x, y+1)$, $\mathfrak{p}_2 = (x, y-1)$ and $\mathfrak{p}_3 = (x,y,z)$. So Y has three components: $L = V(x, y+1), M = V(x, y-1)$ (two lines), and $p = V(x,y,z)$ (the origin). The dimension of Y equals to the largest of the dimension of each component, and $\dim L = 1, \dim M = 1, \dim p = 0$, so

$\dim Y = 1$. The codimension of Y in X equals the maximum of the heights of the associated primes of $\mathfrak{a}$, *i.e.* $\mathrm{ht}(\mathfrak{p}_1) = 2$. So the codimension of Y equals 2. ☆

6.5 Normal schemes and normalization

Definition 6.34 *Let X be an integral scheme with fraction field K. We say that X is* normal *at a point $x \in X$ if the ring $\mathcal{O}_{X,x}$ is integrally closed (viewed as a subring of K).*

Example 6.35 $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ are normal schemes. ☆

Example 6.36 More generally, a scheme which is locally factorial (meaning that all stalks $\mathcal{O}_{X,x}$ are UFD's), is also normal. [CA notes chapter 7]. ☆

For an integral scheme X, we will define a new scheme $\overline{X}$ which is a normal scheme, and a morphism $\pi : \overline{X} \to X$. There are many schemes with this property (take $\operatorname{Spec} K \to X$ for instance), so to get something more canonical, we want $\overline{X}$ and π to satisfy a certain universal property.

We say that a morphism $f : X \to Y$ is *dominant* if the image of f is dense in Y. When X and Y are integral, this is equivalent to saying that the generic point of X maps to the generic point of Y. This means the $f^\sharp$ induces a map between the stalks $f^\sharp \colon \mathcal{O}_{Y,\epsilon} \to \mathcal{O}_{X,\eta}$ where η and ϵ are the generic points in X and Y. But the stalks at the generic points are the function fields $K(X)$ and $K(Y)$; hence we obtain a map $\phi^\sharp \colon K(Y) \to K(X)$, which is injective as any ring map between fields is.

Lemma 6.37 *Let $f : X \to Y$ be a morphism of integral schemes. Then the following are equivalent:*

i) *f is dominant;*

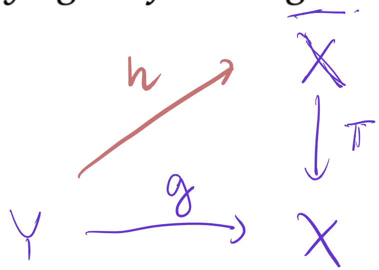
ii) *For all affine open sets $U \subset X$, $V \subset Y$ with $f(U) \subset V$, the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective*

iii) *For one affine open set $U \subset X$, $V \subset Y$ with $f(U) \subset V$, the ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective*

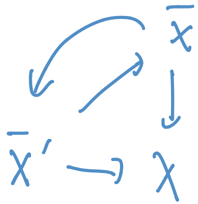
iv) *For all $x \in X$, the local homomorphism $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.*

v) *For one $x \in X$, the local homomorphism $f_x^\sharp : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.*

Theorem 6.38 *Let X be an integral scheme, then there is a normal scheme $\overline{X}$, and a morphism $\pi : \overline{X} \to X$ satisfying the following universal property: For any dominant*



morphism $g : Y \to X$ from a normal scheme Y, there is a unique morphism $h : Y \to \overline{X}$ such that $g = \pi \circ h$.



Proof: The uniqueness part follows from the universal property. We therefore only need to check the existence.

Suppose first that $X = \operatorname{Spec} A$ is affine. Let A' be the normalization of A in the fraction field K.

Let Y be a normal scheme and let $B = \mathcal{O}_Y(Y)$. For a dominant morphism $g : Y \to X$, the map $g^\sharp(X) : A \to B$ is injective, so it factors through a unique morphism $A \to A' \to B$, by the universal property of normalization of rings. Hence g factors via a unique morphism $g' : Y \to \operatorname{Spec} A'$. In particular, the canonical map $\pi : \operatorname{Spec} A' \to \operatorname{Spec} A$ satisfies the universal property in the theorem.

Now let X be an arbitrary integral scheme, and let $U_i = \operatorname{Spec} A_i$ be an affine cover. Note that there are normalization morphisms $\pi_i : U_i' \to U_i$ defined by the inclusions $A_i \subset A_i'$. Consider the open set $U_{ij} = U_i \cap U_j$, which is an open set in both U_i and U_j. As $\pi_i|_{\pi^{-1}(U_{ij}} : \pi^{-1}(U_{ij}) \to U_{ij}$ and $\pi_j|_{\pi^{-1}(U_{ij}} : \pi^{-1}(U_{ij}) \to U_{ij}$ are both normalizations of U_{ij}, they must coincide by the uniqueness. Hence by the Gluing lemma for morphisms, the morphisms π_i glue, so we obtain a scheme X' and a morphism $\pi : X' \to X$. ❑

The X-scheme $\overline{X}$ is called the *normalization* of X.

Corollary 6.39 *The normalization $\overline{X}$ has the following properties:*

i) $\pi : \overline{X} \to X$ *is surjective.*

ii) *There is an open subset $U \subset X$ so that π restricted to $\pi^{-1}(U)$ is an isomorphism.* ⟶ π biragjonal

iii) $\overline{X}$ *and X have the same dimension.*

iv) *If X is of finite type over a field, then $\pi : \overline{X} \to X$ is a finite morphism.*

PROOF: The proof relies on some of the basic properties of the integral closure.

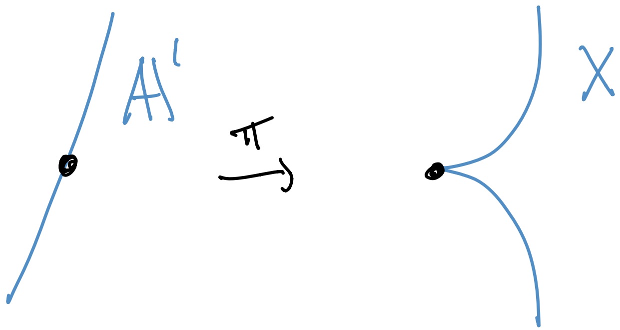
Statement *i)* follows from the Going-Up theorem (or the Lying-Over theorem).

Statement *ii)* holds true because *being normal is a generic property*; that is, for a finitely generated integral domain A, the localization $A_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in U$ in a non-empty open subset U.

Statement *iii)* ensues from the Going-Up theorem.

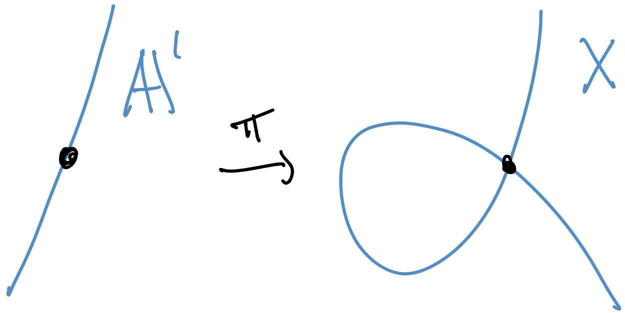
Finally, statement *iv)* follows from the fact that if A is integral domain which is a finitely generated over a field, then the normalization $\widetilde{A}$ in the fraction field K of A is a finite A-module. (This statement is essentially a consequence of the Noether normalization lemma.) ❏

In general, the normalization map $\pi : \overline{X} \to X$ need not be finite in the sense of Section 6.2: Nagata found an example of a local noetherian integral domain A such that the integral closure is not Noetherian (in particular not finite over A). See also Exercise 12.10 in [CA].



Examples

Example 6.40 *(Cuspidal cubic)* Let k be a field, and let $X = \operatorname{Spec} A$ where $A = k[x,y]/(y^2 - x^3)$. This is the *cuspidal cubic curve* in $\mathbb{A}^2_k$. There is an isomorphism of k-algebras $A \xrightarrow{\simeq} k[t^2,t^3]$ given by sending $x \mapsto t^2$ and $y \mapsto t^3$. It is clear that $k[t^2,t^3]$ is an integral domain with fraction field $K = k(t)$. Moreover, the normalization of A equals $\overline{A} = k[t]$. The inclusion $A \subset \overline{A}$ induces the normalization morphism $\pi : \mathbb{A}^1_k \to X$, and this is an isomorphism over the open set $D(t) \subset \mathbb{A}^1_k$ where t is inverible. ☆



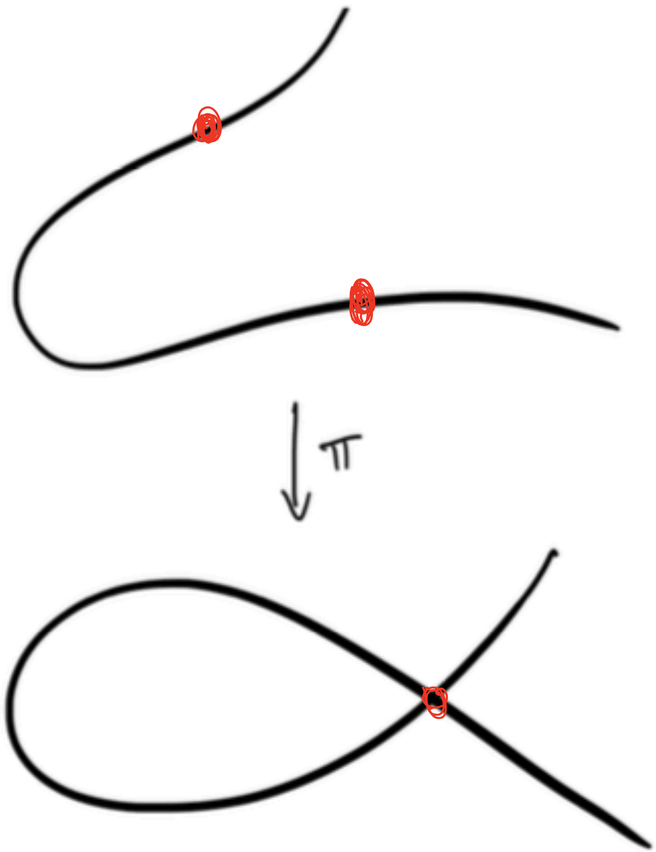
Example 6.41 *(Nodal cubic)* Let now $X = \operatorname{Spec} A$ with A being the ring $A = k[x,y]/(y^2 - x^3 - x^2)$, where k now is a field whose characteristic is not two (if the characteristic is two, we are back in previous cuspidal case). This is the *nodal cubic curve* in $\mathbb{A}_k^2$. Here it is a little bit tricker to find the normalization, but it helps to think about it geometrically.

If we think of the corresponding affine variety $\{ (x, y) \mid y^2 = x^3 + x^2 \} \subset \mathbb{A}^2(k)$, we see that the origin $(0, 0)$ is a special point: a line $l \subset \mathbb{A}^2_k$ through the closed point $(0, 0) \in X$ (with equation $y = tx$) will intersect X at $(0, 0)$ and at one more point (with $x = t^2 - 1$), and this gives a parameterization of the curve, which is generically one-to-one.

Back in the scheme world, we imitate this by introducing the parameter $t = yx^{-1}$ in the function field K of X, the equation $y^2 = x^3 - x^2$ then reduces to $t^2 = 1 + x$ after being divided by x^2. Moreover, the element t is integral, since it satisfies the monic equation $T^2 - x - 1 = 0$ (which has coefficients in A). Since $x = t^2 - 1$ and $y = x \cdot y/x = t^3 - t$, we see that

$$A = k[t^2 - 1, t^3 - t] \subset k[t] \subset K = k(t),$$

and since $k[t]$ is integrally closed, any element in K which is integral over A, can be written as a polynomial in t. So $\overline{A} = k[t]$ is the integral closure of A in $k(t)$. The normalization map $\pi : \operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is an isomorphism outside the origin $(0,0) \in X$. Geometrically the map π identifies two points $(t+1)$ and $(t-1)$ in $\mathbb{A}^1_k$ to the origin in X.


π

Example 6.42 *(The quadratic cone)* Consider the affine scheme $X = \operatorname{Spec} A$ where $A = \mathbb{C}[x,y,z]/(xy - z^2)$. Note that this is not a factorial scheme (A is not a UFD as $xy = z^2$), so we cannot immediately conclude that A is normal. However, there are a few ways to see that it is in fact so:

- There is an isomorphism of rings

$$\phi : A \to \mathbb{C}[u^2, uv, v^2]$$

and the latter algebra is normal in $K = k(u, v)$.

- Let $B = \mathbb{C}[x, y]$, so that $A = B[z]/(z^2 - xy)$. Then $B \subset A$ is a ring extension making A into a finite B-module. We get an inclusion of fields $K(B) = \mathbb{C}(x, y) \subset K(A)$ obtained by adjoining the element z $(= \sqrt{xy})$. Write an element of $K(A)$ as $w = u + v$ where $u, v \in K(B) = \mathbb{C}(x, y)$. If this is integral over A, it is also integral over B. In fact, w satisfies the minimal polynomial

$$T^2 - 2uT - (x^2 + y^2)v^2 = 0$$

If this is integral over B, we must have $2u \in \mathbb{C}[x, y]$ and hence $u \in \mathbb{C}[x, y]$. Moreover $u^2 - (x^2 + y^2)v^2 \in \mathbb{C}[x, y]$, so also $(x^2 + y^2)v^2 \in k[x, y]$. Note that $(x^2 + y^2) = (x - iy)(x + iy)$ is a product of coprime, and irreducible elements, so we must have also $v^2 \in k[x, y]$, and for the same reason $v \in k[x, y]$. Hence $u + vz \in B[z]$.