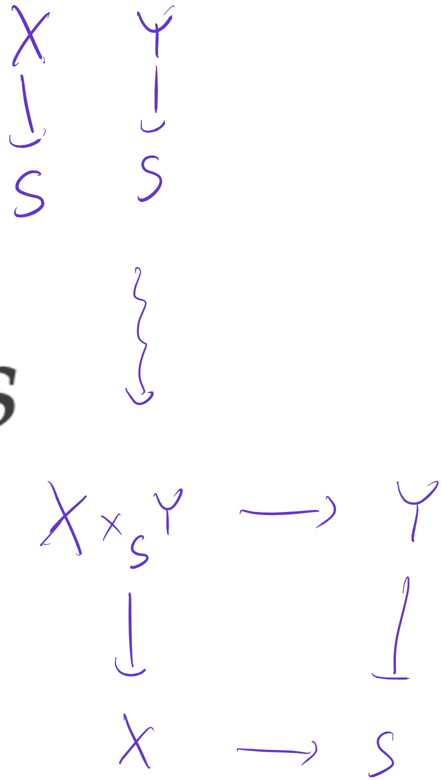
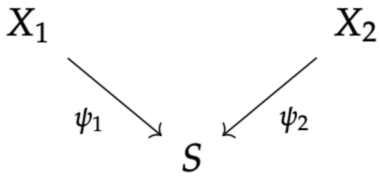
Chapter 7
Fibre products



Fibre products of sets.

As a warming up we use some lines on recalling the fibre product in the category Sets of sets. The points of departure is two sets X_1 and X_2 both equipped with a map to a third set S; *i.e.* we are given a diagram



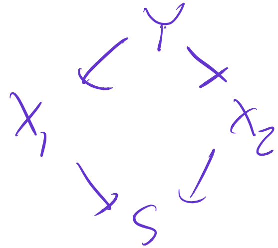
The fibre product $X_1 \times_S X_2$ is the subset of the cartesian product $X_1 \times X_2$ consisting of the pairs whose two components have the same image in S; that is, we have

$$X_1 \times_S X_2 = \{ (x_1, x_2) \mid \psi_1(x_1) = \psi_2(x_2) \}.$$

Clearly the diagram below where π_1 and π_2 denote the restrictions of the two projections to the fibre product; in other words, $\pi_i(x_1, x_2) = x_i$, is commutative,

$$\begin{array}{ccccc} & & X_1 \times_S X_2 & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ X_1 & & & & X_2 \\ & \searrow_{\psi_1} & & \swarrow_{\psi_2} & \\ & & S & & \end{array} \tag{7.1}$$

And more is true: the fibre product enjoys a universal property. Given any two maps $\phi_1 \colon Z \to X_1$ and $\phi_2 \colon Z \to X_2$ such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ there is a unique map $\phi \colon Z \to X_1 \times_S X_2$ satisfying $\pi_1 \circ \phi = \phi_1$ and $\pi_2 \circ \phi = \phi_2$. To lay your hands on such a ϕ, we just use the map whose two components are ϕ_1 and ϕ_2 and observe that it takes values in $X_1 \times_S X_2$ since the relation $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ holds.



The fibre product in general categories

The notion of a fibre product—formulated as the solution to a universal problem as above—is *mutatis mutandis* meaningful in any category C. Given any two arrows $\psi_i \colon X_i \to S$ in the category C, an object—that we shall denote by $X_1 \times_S X_2$—is said to be the *fibre product* (fiberproduktet) of the objects X_i, or more precisely of the two arrows $\psi_i \colon X_i \to S$, if the following two conditions are fulfilled:

- There are two arrows $\pi_i \colon X_1 \times_S X_2 \to X_i$ in C such that $\psi_1 \circ \pi_1 = \psi_2 \circ \pi_2$ (called the projections).
- For any two arrows $\phi_i \colon Z \to X_i$ in C such that $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$, there is a *unique* arrow $\phi \colon Z \to X_1 \times_S X_2$ satisfying $\pi_i \circ \phi = \phi_i$ for $i = 1, 2$.

The two arrows $\pi_1 \circ \phi$ and $\pi_2 \circ \phi$ that determine the arrow $\phi\colon Z \to X_1 \times_S X_2$, are called *the components* (komponentene) of ϕ, and the notation $\phi = (\phi_1, \phi_2)$ is sometimes used. If $\phi_1\colon Y_1 \to X_1$ and $\phi_2\colon Y_2 \to X_2$ are two arrows over S, there is a unique arrow denoted $\phi_1 \times \phi_2$ from $Y_1 \times_S Y_2$ to $X_1 \times_S X_2$ whose components are $\phi_1 \circ \pi_{Y_1}$ and $\phi_2 \circ \pi_{Y_2}$.

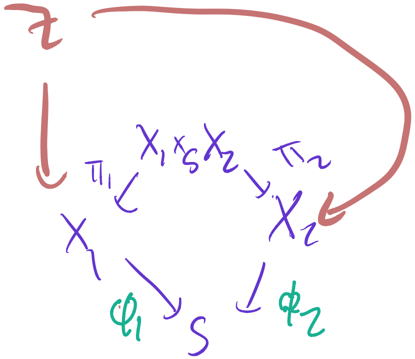
It is not so hard to come up with examples of categories where fibre products do *not* exist. For instance, consider the category C where the objects are subsets X of the integers with an even number of elements, and the morphisms given by inclusions $Y \subset X$. In this category, the fibre product of $Y \subset X$ and $Z \subset X$ over X would be $Y \cap Z \subset X$. However, $Y \cap Z$ does not necessarily have an even number of elements!

Theorem 7.1 (Existence of fibre products) *Let $X \to S$ and $Y \to S$ be two schemes over the scheme S. Then their fibre product $X \times_S Y$ exists.*

The projections onto X and Y will frequently be denoted by respectively π_X and π_Y. We will see several examples later which show that the underlying set of a product can be very different from the product of the underlying sets of X and Y. However, the 'scheme-valued points' behave well; that is, for any S-scheme $T \to S$, there is a canonical isomorphism of sets of T-points

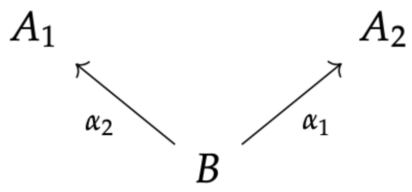
$$\mathrm{Hom}_{\mathsf{Sch}/\mathsf{S}}(T, X \times_S Y) \simeq \mathrm{Hom}_{\mathsf{Sch}/\mathsf{S}}(T, X) \times \mathrm{Hom}_{\mathsf{Sch}/\mathsf{S}}(T, Y),$$

which is just another way of formulating the universal property of a product.

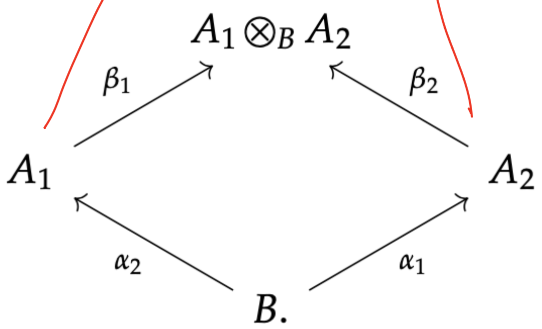


7.2 *Products of affine schemes*

The category AffSch of affine schemes is, more or less by definition, equivalent to the category of rings, and in the category of rings we have the tensor product. The tensor product enjoys a universal property *dual* to the one of the fibre product. To be precise, assume A_1 and A_2 are B-algebras, *i.e.* we have two maps of rings α_i



There are two maps $\beta_i \colon A_i \to A_1 \otimes_B A_2$ sending $a_1 \in A_1$ to $a_1 \otimes 1$ and a_2 to $1 \otimes a_2$, respectively. These are both ring homomorphisms since $aa' \otimes 1 = (a \otimes 1)(a' \otimes 1)$ respectively $1 \otimes aa' = (1 \otimes a)(1 \otimes a')$, and they fit into the commutative diagram



(7.2)

because $\alpha_1(b) \otimes 1 = 1 \otimes \alpha_2(b)$ by the definition of the tensor product $A_1 \otimes_B A_2$

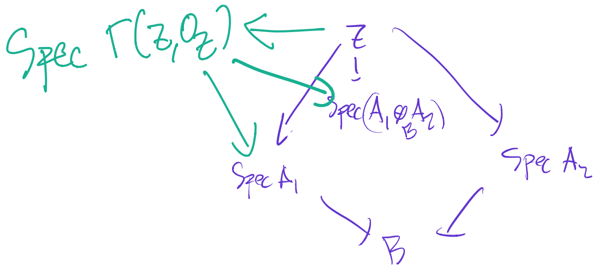
Moreover, the tensor product is *universal* in this respect. Indeed, assume that $\gamma_i \colon A_i \to C$ are B-algebra homomorphisms, *i.e.* $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$; or said differently, they fit into the commutative diagram analogous to (7.2) with the β_i's replaced by the γ_i's. The association $a_1 \otimes a_2 \to \gamma_1(a_1)\gamma(a_2)$ is B-bilinear, and hence it extends to a B-algebra homomorphism $\gamma \colon A_1 \otimes_B A_2 \to C$, which obviously has the property $\gamma \circ \beta_i = \gamma_i$.

Applying the Spec-functor to all this, we get the diagram

$$\begin{array}{ccc} & \operatorname{Spec}(A_1 \otimes_B A_2) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \operatorname{Spec} A_1 & & \operatorname{Spec} A_2 \\ \searrow & & \swarrow \\ & \operatorname{Spec} B & \end{array} \tag{7.3}$$

and the affine scheme $\operatorname{Spec}(A_1 \otimes_B A_2)$ enjoys the property of being universal among affine schemes sitting in a diagram like (7.3). Hence $\operatorname{Spec}(A_1 \otimes_B A_2)$ equipped with the two projections π_1 and π_2 is the fibre product in the category AffSch of affine schemes. One even has the stronger statement; it is the fibre

Proposition 7.2 *Given* $\phi_i \colon \operatorname{Spec} A_i \to \operatorname{Spec} B$*. Then* $\operatorname{Spec}(A_1 \otimes_B A_2)$ *with the two projection* π_1 *and* π_2 *defined as above, is the fibre product of the* $\operatorname{Spec} A_i$*'s in the category of schemes. That is, if* Z *is a scheme and* $\psi_i \colon Z \to \operatorname{Spec} A_i$ *are morphisms with* $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$*, there exists a unique morphism* $\psi \colon Z \to \operatorname{Spec}(A_1 \otimes_B A_2)$ *such that* $\pi_i \circ \psi = \psi_i$ *for* $i = 1, 2$*.*



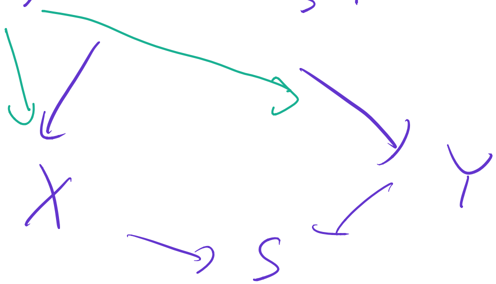
Proof: We know that the proposition is true whenever Z is an affine scheme; so the salient point is that Z is not necessarily affine. For short, we let $X = \operatorname{Spec}(A_1 \otimes_B A_2)$. The proof is just an application of the gluing lemma for morphisms. One covers Z by open affines U_α and covers the intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by open affine subsets $U_{\alpha\beta\gamma}$ as well. By the affine case of the proposition, for each U_α we get a map $\psi_\alpha \colon U_\alpha \to X$, such that $\pi_i \circ \psi_\alpha = \psi_i|_{U_\alpha}$. By the uniqueness part of the affine case, these maps coincide on the open affines $U_{\alpha\beta\gamma}$, and therefore on the intersections $U_{\alpha\beta}$. They can thus be patched together to a map $\psi \colon Z \to X$, which is is unique since the ψ_α's are unique. ❑

A useful lemma

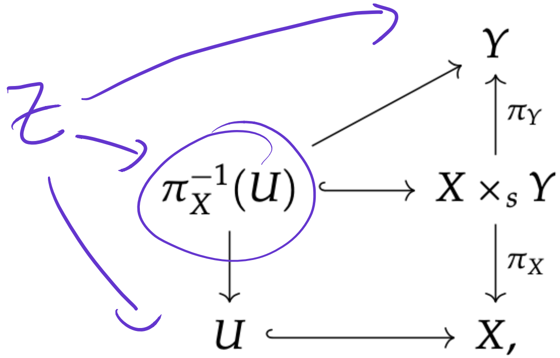
Recall that any open subset U of a scheme Y has a canonically defined scheme structure as an open subscheme; the structure sheaf equals $\mathcal{O}_Y|_U$. Hence, if f is any morphism $f\colon X \to Y$, the inverse image $f^{-1}(U)$ is in a natural way an open subscheme of X. The following lemma will turn out to useful:

Lemma 7.3 *If $X \times_S Y$ exists and $U \subset X$ is an open subscheme, then $U \times_S Y$ exists and is (canonically isomorphic to) an open subset of $X \times_S Y$, moreover projections restrict to projections. Indeed, $\pi_X^{-1}(U)$ with the two restrictions $\pi_Y|_{\pi_X^{-1}(U)}$ and $\pi_X|_{\pi_X^{-1}(U)}$ as projections is the fibre product $U \times_S Y$.*

$$U \times_S Y = \pi_X^{-1}(U) \subset X \times_S Y$$



PROOF: Displayed the situation appears like



and we are to verify that $\pi_X^{-1}(U)$ together with the restriction of the two projections to $\pi_X^{-1}(U)$ satisfy the universal property. If Z is a scheme and $\phi_U \colon Z \to U$

tions to $\pi_X^{-1}(U)$ satisfy the universal property. If Z is a scheme and $\phi_U \colon Z \to U$ and $\phi_Y \colon Z \to Y$ are two morphisms over S we may consider ϕ_U as a map into X, and therefore they induce a map of schemes $\phi \colon Z \to X \times_S Y$ with $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. Clearly $\pi_X \circ \phi = \phi_U$ takes values in U and therefore ϕ takes values in $\pi_X^{-1}(U)$. It follows immediately that ϕ is unique (see the exercise below), and we are through. ❑

When identifying $\pi_X^{-1}(U)$ with $U \times_S Y$, the inclusion map $\pi_X^{-1}(U) \subset X \times_S Y$ will correspond to the map $\iota \times \mathrm{id}_Y$ where $\iota \colon U \to X$ is the inclusion, so a reformulation of the lemma is that open immersions stay open immersions under change of basis.

The following proposition will be basis for all gluing necessary for the construction:

Proposition 7.4 *Let $\psi_X \colon X \to S$ and $\psi_Y \colon Y \to S$ be two maps of schemes, and assume that there is an open covering $\{U_i\}_{i \in I}$ of X such that $U_i \times_S Y$ exist for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open covering of $X \times_S Y$ and projections restrict to projections.*

An immediate consequence of the gluing proposition 7.4 is that fibre products exist over an affine base S.

Lemma 7.5 *Assume that S is affine, then $X \times_S Y$ exists.*

Proof: First if Y as well is affine, we are done. Indeed, cover X by open affine sets U_i. Then $U_i \times_S Y$ exists by the affine case, and we are in the position to apply proposition 7.4 above. We then cover Y by affine open sets V_i. As we just verified, the products $X \times_S V_i$ all exist, and applying proposition 7.4 once more, we can conclude that $X \times_S Y$ exists. ❑

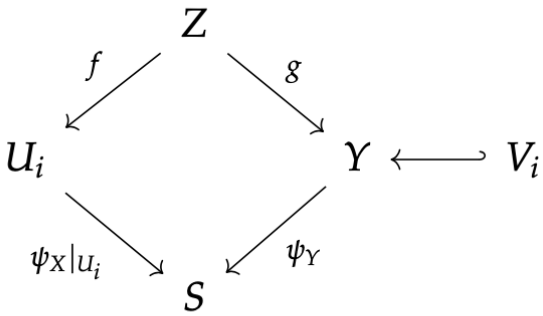
7.4 The final reduction

Let $\{S_i\}$ be an open affine covering of S and let $U_i = \psi_X^{-1}(S_i)$ and $V_i = \psi_Y^{-1}(S_i)$. By Lemma 7.5 the products $U_i \times_{S_i} V_i$ all exist. Using the following Lemma and, for the third time, the gluing Proposition 7.4 we are through with a proof of the existence of fibre products (Theorem 7.1 on page 127).

Lemma 7.6 *With current notation, we have the equality $U_i \times_{S_i} V_i = U_i \times_S Y$. That is, $U_i \times_S Y$ exists and the projections are π_{U_i} and $\pi_Y|_{V_i}$.*

Lemma 7.6 *With current notation, we have the equality* $U_i \times_{S_i} V_i = U_i \times_S Y$*. That is,* $U_i \times_S Y$ *exists and the projections are* π_{U_i} *and* $\pi_Y|_{V_i}$*.*

Proof: We contend that $U_i \times_{S_i} V_i$ satisfies the universal product property of $U_i \times_S Y$. The key diagram is

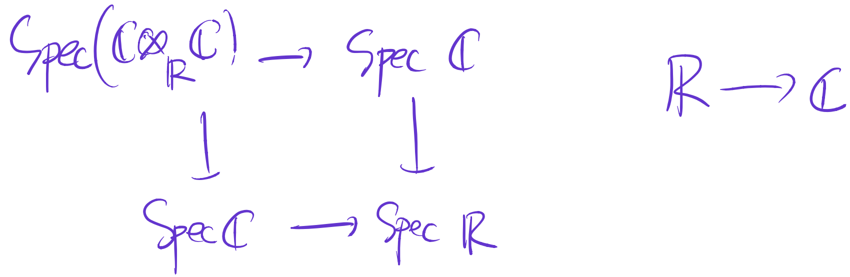


where f and g are two given maps. If one follows the left path in the diagram, one ends up in S_i, and hence the same must hold following the right path. But then, V_i being equal to the inverse image $\psi_Y^{-1}(S_i)$, it follows that g factors through V_i, and by the universal property of $U_i \times_{S_i} V_i$ there is a morphism $Z \to U_i \times_{S_i} V_i$ with the requested properties. ❑

Diagrams arising from fibre products are frequently called *Cartesian diagram* (kartesiske diagrammer); that is, the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Y}\big\downarrow & & \big\downarrow{\scriptstyle \psi_X} \\ Y & \xrightarrow[\psi_Y]{} & S \end{array}$$

is said to be a *Cartesian diagram* if there is an isomorphism $Z \simeq X \times_S Y$ with π_X and π_Y corresponding to the two projections.



Examples

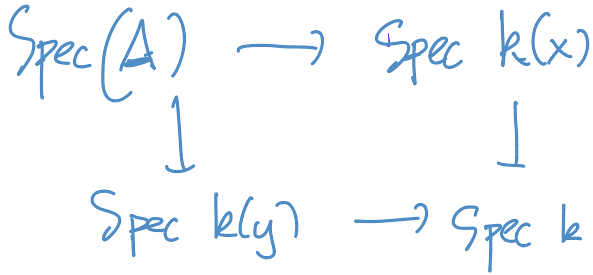
7.7 A simple but illustrative example is the product $\operatorname{Spec}\mathbb{C} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$. This scheme has *two* distinct closed points, and it is not integral—it is not even connected!

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is in fact isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$ of two copies of the complex field $\mathbb{C}$; indeed, we compute using that $\mathbb{C} = \mathbb{R}[t]/(t^2+1)$ and find

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2+1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2+1) = \mathbb{C}[t]/(t-i)(t+i) = \mathbb{C} \times \mathbb{C}$$

where for the last equation we use the Chinese remainder theorem and that the rings $\mathbb{C}[t]/(t \pm i)$ both are isomorphic to $\mathbb{C}$.

The example also shows that the underlying set of the fibre product is not necessarily equal to the fibre product of the underlying sets, although this was true for varieties over an algebraically closed field. In the present case the three schemes involved all have just one element and the their fibre product has just one point. So we issue warnings: The product of integral schemes is in general not necessarily integral! The underlying set of the fibre product is not always the fibre product of the underlying sets.



7.9 Let k be a field and x and y two variables. Consider the tensor product $A = k(x) \otimes_k k(y)$. We can regard this as a localization of $k[x, y]$ where we invert everything in the multiplicative set $S = \{p(x)q(y) | p(x), q(y) \neq 0\}$. Let us show that A has infinitely many maximal ideals. Suppose that $\mathfrak{m} \subset A$ is a maximal ideal; it has the form $S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset k[x, y]$ which is maximal among the primes that do not intersect S. In this case we must have $\mathfrak{p} \cap k[x] = 0$, since otherwise there would be a non-zero $p(x) \in \mathfrak{p} \cap S$. Similarly $\mathfrak{p} \cap k[y] = 0$, which implies that $\mathfrak{p}$ has height at most 1. Hence either $\mathfrak{p} = (0)$, of $\mathfrak{p} = (f)$ for some irreducible polynomial $f \in k[x, y]$ not a product of a polynomial from $k[x]$ and one from $k[y]$. If follows that A has dimension 1, and A has infinitely many maximal ideals—in fact uncountably many if *e.g.* $k = \mathbb{C}$.

This example shows how strange the fibre product really is— Spec A is an infinite set, even though it is the fibre product of two schemes with one-point underlying sets. We will see more examples like this in the end of this chapter.

ex X, Y affine varieteter

end. k-algebra + int. omr.

$\rightsquigarrow X \times_k Y = \operatorname{Spec}(A(X) \otimes_k A(Y))$

7.7 *Scheme theoretic fibres*

In most parts of mathematics, when one studies a map of some sort, a knowledge of what the fibres of the map are, is of great help. This is also true in the theory of schemes.

Suppose that $\phi\colon X \to Y$ is a map of schemes and that $y \in Y$ is a point. On the level of topological spaces, we are interested in the preimage $\phi^{-1}(y)$, and we aim at giving a scheme theoretic definition of the fibre $\phi^{-1}(y)$. Having the

$=: X_y$

$$\begin{array}{ccc}
\phi^{-1}(y) = X \times_Y \operatorname{Spec} k(y) & \longrightarrow & X \\
\downarrow & & \downarrow \phi \\
\operatorname{Spec} k(y) & \longrightarrow & Y.
\end{array}$$

As the next lemma will show, the underlying topological space of $\phi^{-1}(y)$ is the topological fibre, but additionally it is endowed with a scheme structure. In many cases it will not be reduced, and this a mostly a good thing since it makes certain continuity results true. One often writes X_y for the fibre $\phi^{-1}(y)$.

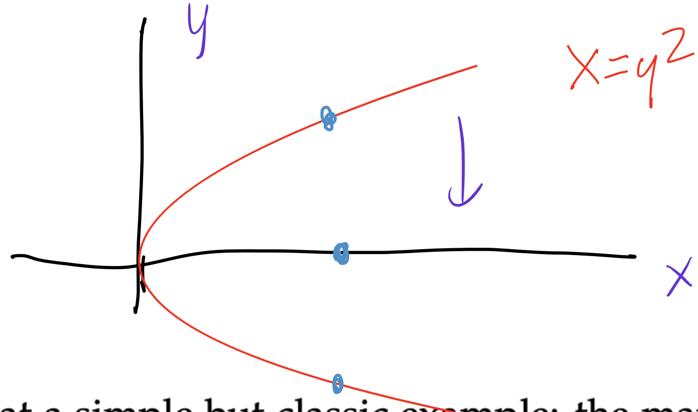
Proposition 7.13 *Let X and Y be a schemes and $\phi\colon X \to Y$ a morphism. Let $y \in X$ be a point.*

Y

i) *The inclusion $X_y \to X$ of the scheme theoretic fibre is a homeomorphism onto the topological fibre $\phi^{-1}(y)$;*

ii) *If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, it holds that $X_y = \operatorname{Spec}(A/\mathfrak{p}_y A)_{\mathfrak{p}_y}$;*

iii) *If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and y is a closed point, one has $X_y = \operatorname{Spec} A/\mathfrak{m}_y A$.*



Example 7.14 We take a look at a simple but classic example: the map

$$\phi\colon \operatorname{Spec} k[x,y]/(x-y^2) \to \operatorname{Spec} k[x]$$

induced by the injection $B = k[x] \to k[x,y]/(x-y^2) = A$. Geometrically one would say it is just the projection of the parabola onto the x-axis.

If $a \in k$, computing the fibre over $\mathfrak{m}_a = (x - a)$ yields, that $\phi^{-1}(\mathfrak{m}_a)$ is the spectrum of the ring

$$k[x,y]/(x - y^2) \otimes_{k[x]} k(a) \simeq k[y]/(y^2 - a).$$

where $k(a)$ denotes the field $k(a) = k[t]/(t - a)$ (which of course is just a copy of k), and where we are using the isomorphism $R/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M$ for an ideal $\mathfrak{a}$ in an A-module M.

Several cases can occur, apart from the characteristic two case which is special.

i) If a does not have a square root in k, the fiber is $\operatorname{Spec} k(\sqrt{a})$ where $k(\sqrt{a})$ is a quadratic field extension of k.

ii) In case a has a square root in K, way $b^2 = a$, the polynomial $y^2 - a$ factors as $(y-b)(y+b)$, and the fibre becomes the product

$$\operatorname{Spec} k[y]/(y-b) \times \operatorname{Spec} k[y]/(y+b),$$

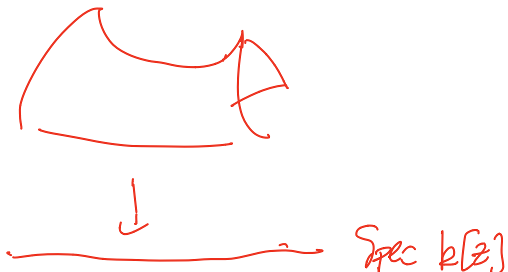
which is the disjoint union of two copies of $\operatorname{Spec} k$

iii) The final case appears when $a = 0$. The the fibre is not reduced, but equals $\operatorname{Spec} k[y]/y^2$.

↖ fiberprodukt av to reduerte kan bli ikke-redusert.

We also notice that the generic fibre of ϕ is the quadratic extension $k(x)(\sqrt{x})$ of the function field $k(x)$.

Over perfect fields k of characteristic two, the picture is completely different. Then a is a square, say $a = b^2$ and as $(y^2 - b^2) = (y - b)^2$ non of the fibers are reduced, they equal $\operatorname{Spec} k[y]/(y - b)^2$, except the generic one which is $k(x)(\sqrt{x})$. One observes interestingly enough, that all the non-reduced fibres deform into a field! ☆

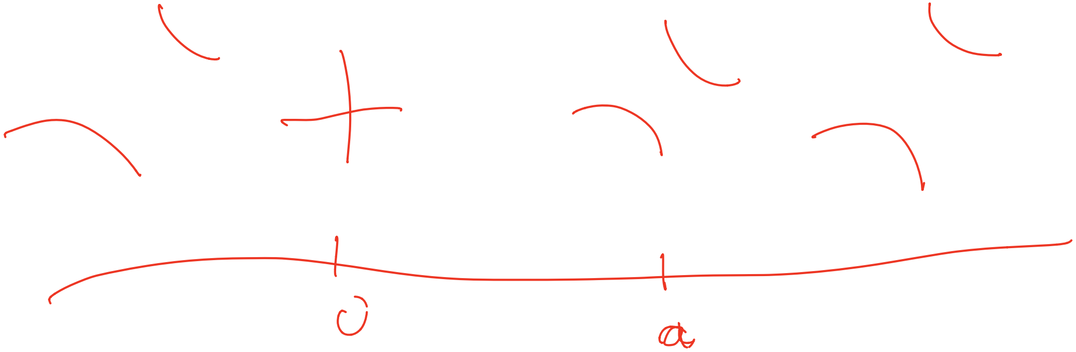


Example 7.15 A similar example can be obtained from the map

$$f : \operatorname{Spec} A \to \operatorname{Spec} B$$

where $A = \operatorname{Spec} k[x, y, z]/(xy - z)$ and $B = k[z]$ and f is induced from the obvious inclusion $k[z] \to k[x, y, z]/(xy - z)$. As before, we assume k algebraically closed, pick a closed point $a \in \operatorname{Spec} B$, and consider the fibre

$$X_a = \operatorname{Spec}(A \otimes_B k(a)) = \operatorname{Spec} k[x, y]/(xy - a)$$



Again, two cases occur. If $a \neq 0$, in which case $xy - a$ is an irreducible polynomial, and so X_a is an integral scheme. This is intuitive, since it corresponds to the hyperbola $\{xy = a\}$. If $a = 0$, we are left with $X_0 = \operatorname{Spec} k[x, y]/(xy)$, which is not irreducible; it has two components corresponding to $V(x)$ and $V(y)$. (X_0 is reduced however).

For good measure, we also consider the fibre over the generic point η of $\operatorname{Spec} B$. This corresponds to

$$k[x,y,z]/(xy-z)\otimes_{k[z]} k(z) = k(z)[x,y]/(xy-z)$$

which is an integral domain. Hence X_η is integral. ☆