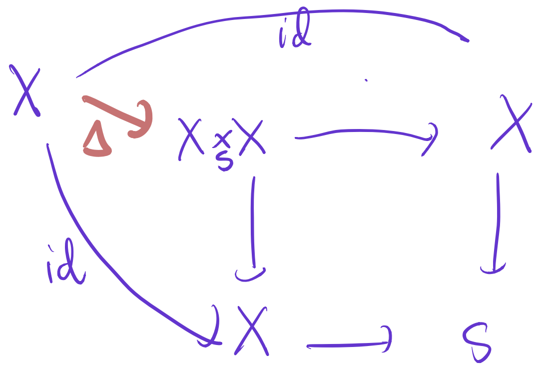
Chapter 8

Separated schemes

Hausdorff aksiomet fra Alg. geo I.

The topology on schemes behaves very differently from the usual Euclidean topology. In particular, schemes are not Hausdorff, except in trivial cases – the open sets in the Zariski topology are simply too large. Still we would like to find an analogous property that can serve as a satisfactory substitute for this property. The route we take is to impose that the diagonal should be closed; closed in the Zariski topology of the product, of course.



8.1 The diagonal

Let X/S be a scheme over S. There is a canonical map $\Delta_{X/S}\colon X \to X \times_S X$ of schemes over S called the *diagonal map* or the *diagonal morphism*. The two component maps of $\Delta_{X/S}$ are both equal to the identity id_X; that is, the defining properties of $\Delta_{X/S}$ are $\pi_i \circ \Delta_{X/S} = \mathrm{id}_X$ for $i = 1, 2$ where the π_i's denote the two projections.

$\operatorname{Spec} A \to \operatorname{Spec} B \iff B \to A \to 0$

In the case that X and S are affine schemes, the diagonal has a simple and natural interpretation in terms of algebras; it corresponds to the most natural map, namely the multiplication map:

$$\mu \colon A \otimes_B A \to A.$$

$a \otimes a' \mapsto aa'$

$\rightsquigarrow$ surjektiv ✓

$\rightsquigarrow$ $\operatorname{Spec} A \longrightarrow \operatorname{Spec}(A \otimes_B A)$ er en lukket immersjon

The multiplication map sends $a \otimes a'$ to the product aa' and then extends to $A \otimes_B A$ by linearity. The projections correspond to the two algebra homomorphisms $\iota_i \colon A \to A \otimes_B A$ sending a to $a \otimes 1$ respectively to $1 \otimes a$. Clearly it holds that $\mu \circ \iota_i = \mathrm{id}_A$, and on the level of schemes this translates into the defining relations for the diagonal map. Moreover, μ is clearly surjective, so we have established the following:

Proposition 8.1 *If X an affine scheme over the affine scheme S, then the diagonal $\Delta_{X/S}\colon X \to X \times_S X$ is a closed immersion.*

The conclusion here is not generally true for schemes, and shortly we shall give counterexamples. However from the proposition we just proved, it follows readily that the image $\Delta_{X/S}(X)$ is always *locally closed*, *i.e.* the diagonal is locally a closed immersion:

Proposition 8.2 *The diagonal $\Delta_{X/S}$ is locally a closed immersion.*

Proof: Begin with covering S by open affine subsets and subsequently cover each of their inverse images in X by open affines as well. In this way one obtains a covering of X by affine open subsets U_i whose images in S are contained in affine open subsets S_i. The products $U_i \times_{S_i} U_i = U_i \times_S U_i$ are open and affine, and their union is an open subset containing the image of the diagonal. By Proposition 8.1 above the diagonal restricts to a closed immersion of U_i in $U_i \times_{S_i} U_i$. ❑

8.2 Separated schemes

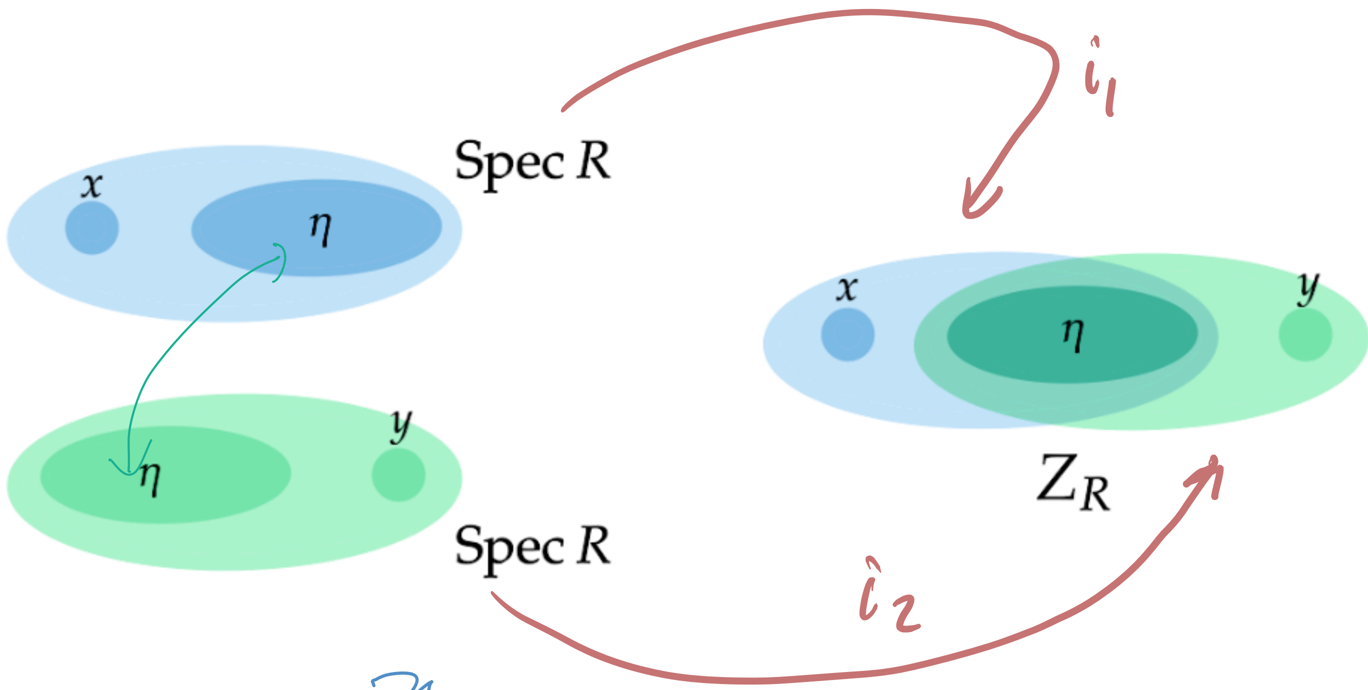
We have now come to the definition of the property that will play the role of the Hausdorff property for schemes.

Definition 8.3 *One says that the scheme X/S is* separated *over S, or that the structure map $X \to S$ is separated, if the diagonal map $\Delta_{X/S} : X \to X \times_S X$ is a closed immersion. One says for short that X is separated if it is separated over* $\operatorname{Spec} \mathbb{Z}$.

Examples

8.4 Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is separated (by Proposition 8.1)

8.5 The simplest example of a scheme that is not separated, is obtained by glueing the prime spectrum of a discrete valuation ring to itself along the generic point.



$R = \mathbb{Z}_{(2)}$

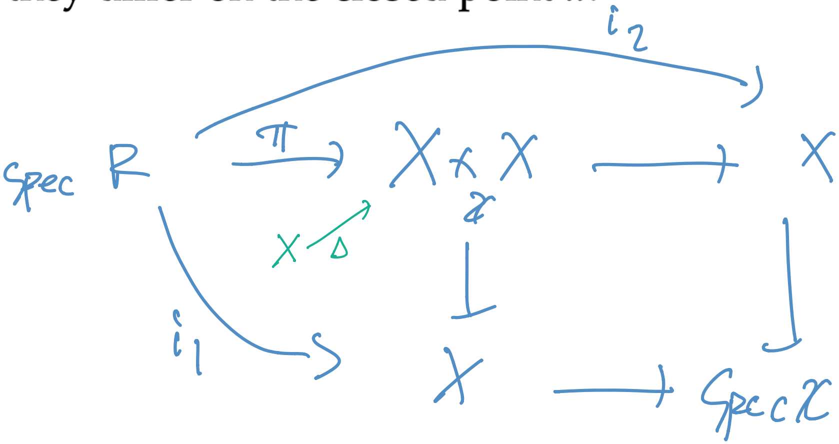
$k[x]_{(x)}$

$$\pi^{-1}(\Delta) = \{ x \in \operatorname{Spec} R \mid i_1(x) = i_2(x) \}$$

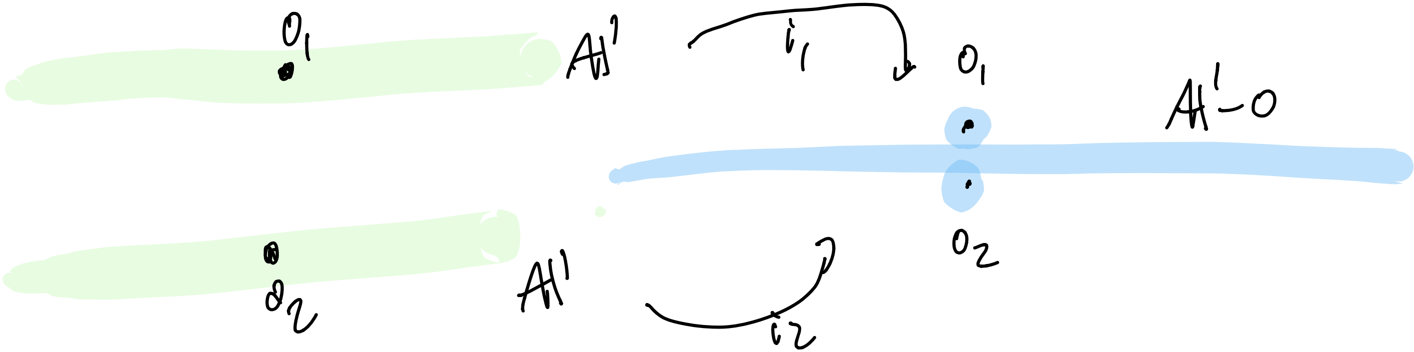
$\Delta(X)$

In this manner we construct a scheme Z_R together with two open immersions $\iota_i\colon \operatorname{Spec} R \to Z_R$. They send the generic point η to the same point, which is an open point in Z_R, but they differ on the closed point x.

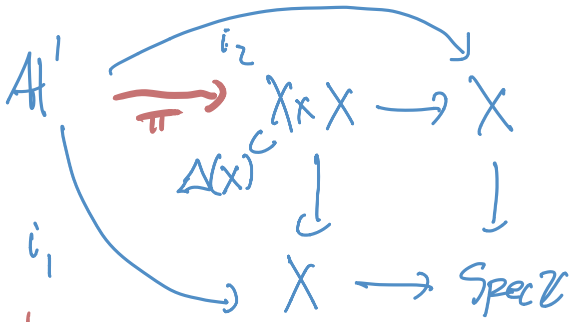
$X = Z_R$



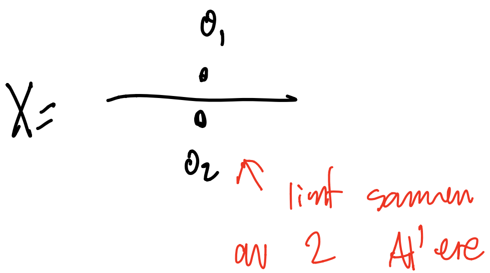
It follows that the diagonal is not closed. Indeed, the subspace of Spec R where the two maps ι_i agree is the preimage of the diagonal. But this subspace has exactly one point, the open point, which is not closed.

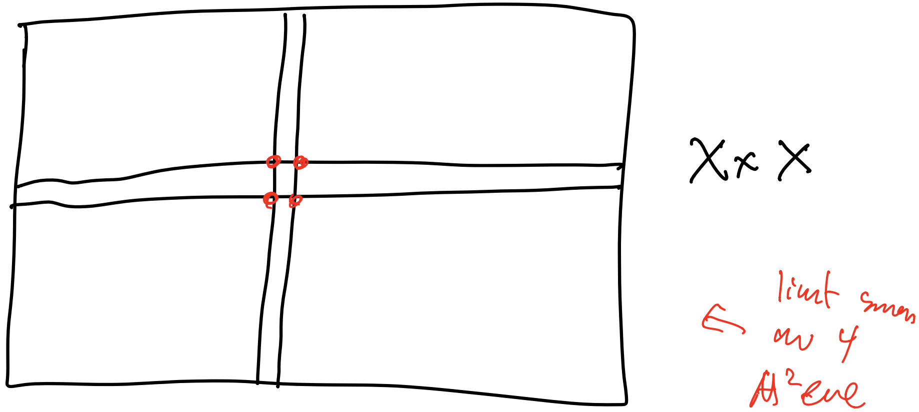


8.6 The affine line X with two origins constructed on page 96 in Chapter 5 is not separated over $S = \operatorname{Spec} A$. It was constructed as the union of two affine lines $U_i = \operatorname{Spec} A[u]$ glued together along their common open subset $U_{12} = \operatorname{Spec} A[u, u^{-1}]$. Hence there are two open immersions $\operatorname{Spec} A[u] \to X$ which agree on U_{12} which is not closed, and according to Proposition 8.14 below, it can not be separated.



$$\pi^{-1}(\Delta(X)) = \{ x \in \mathbb{A}^1 \mid i_1(x) = i_2(x) \} = \mathbb{A}^1 - 0 = U_{12}$$





It is also instructive to examine the diagonal in detail. Denote the two origins by O_1 and O_2. Then the scheme $X \times_A X$ is an affine plane with double coordinate axes, and four origins $(O_1, O_1), (O_1, O_2), (O_2, O_1), (O_2, O_2)$. However, the image of the diagonal morphism only contains the two origins (O_1, O_1) and (O_2, O_2) while the closure of $\Delta_{X/S}(X)$ contains all four origins. ☆

Separerte skjemaer har veldig bra egenskaper:

$$\mathbb{C} \longrightarrow A \otimes_R \mathbb{C}$$
$$R \longrightarrow A$$

$$\begin{array}{ccc} \operatorname{Spec}(A \otimes_R \mathbb{C}) = X_{\mathbb{C}} & \longrightarrow & X = \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} R \end{array}$$

Proposition 8.9 *The following hold true:*

i) *Locally closed immersions are separated, in particular open and closed immersions are;*

ii) *A composition of two separated morphisms is again separated;*

iii) *Separatedness is stable under base change: if $f : X \to S$ is separated, and $T \to S$ is any morphism, then $f_T : X \times_S T \to T$ is separated.*

$$\begin{array}{ccc} X \times_S T & \longrightarrow & X \\ \Rightarrow \text{separert} \downarrow & & \downarrow \text{separert} \\ T & \longrightarrow & S \end{array}$$

By the immense freedom we have for gluing schemes together, there are lots of non-separated schemes in the world of schemes. On the other hand, the examples above are obviously a bit peculiar, and one doesn't frequently encounter non-separated schemes in practice. In fact, the first edition of EGA reserved the word *preschema* for what we today call schemes, and *schema* for a separated scheme.

More importantly, some very nice properties hold only for separated schemes, and this legitimates the notion. Of course, one needs good criteria to be sure we have a large class of separated schemes. We have already seen that all affine schemes are separated, and we will see in Chapter 9 that the same is true for projective schemes also.

One of the nice properties separated schemes enjoy, is the following:

Proposition 8.10 *Assume that X is separated and that U and V are two affine open sets. Then the intersection $U \cap V$ is also affine and the natural product map $\Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V) \rightarrow \Gamma(U \cap V, \mathcal{O}_X)$ is surjective*

PROOF: The product $U \times V$ is an open and affine subset of $X \times V$, and $U \cap V = \Delta_X(X) \cap (U \times V)$. So if the diagonal is closed, $U \cap V$ is a closed subset of the affine set $U \times V$ hence affine (Proposition 3.19). By the construction of the fiber product of affine schemes that one has

$$\Gamma(U \times V, \mathcal{O}_{U\times V}) = \Gamma(U, \mathcal{O}_U) \otimes \Gamma(V, \mathcal{O}_V),$$

and as $U \cap V$ is a closed subscheme of $U \times V$, the restriction map

$$\Gamma(U \times V, \mathcal{O}_{U\times V}) \to \Gamma(U \cap V, \mathcal{O}_{U\cap V})$$

is surjective, as we wanted to show. ❑

Conversely, we have

Proposition 8.11 *Let X be a scheme, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine cover such that*

i) all intersections $U_i \cap U_j$ are affine;

ii) $\Gamma(U_i, \mathcal{O}_X) \otimes \Gamma(U_j, \mathcal{O}_X) \to \Gamma(U_i \cap U_j, \mathcal{O}_X)$ is surjective for each $i, j \in I$.

Then X is separated.

PROOF: Let $\pi_1, \pi_2 : X \times X \to X$ be the two projections and let $\Delta : X \to X \times X$ denote the diagonal morphism. Let $U_i = \operatorname{Spec} B_i$ and $U_j = \operatorname{Spec} B_j$ be two open sets in the covering $\mathcal{U}$. We have

$$\Delta^{-1}(\pi_1^{-1}(U_i) \cap \pi_2^{-1}(U_j)) = \Delta^{-1}(\pi_1^{-1}(U_i)) \cap \Delta^{-1}(\pi_2^{-1}(U_j))) = U_i \cap U_j \quad (8.2)$$

Also, from the universal property of the fibre product, we get that $\pi_1^{-1}(U_i) \cap \pi^{-1}(U_j) = U_i \times U_j \subset X \times X$. From this we deduce that Δ is a closed immersion if each of the restrictions of Δ

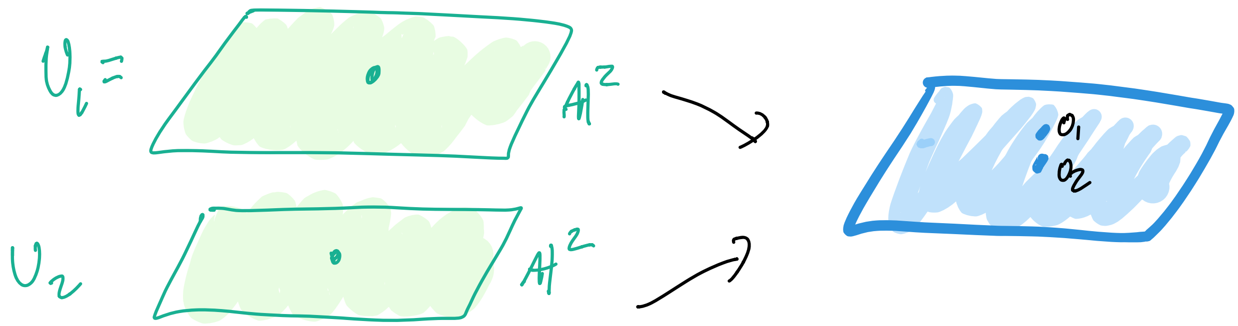
$$\Delta_{ij} : U_i \cap U_j \to U_i \times U_j$$

is a closed immersion. But this follows from the assumptions: by *i)*, the intersection $U_i \cap U_j$ is affine, say $U_i \cap U_j = \operatorname{Spec} C_{ij}$, and by *ii)*, the ring homomorphism $B_i \otimes_A B_j \to C_{ij}$ is surjective. Hence Δ_{ij} is a closed immersion for each i, j, and the proof is complete. ❑

Example 8.12 The projective line $\mathbb{P}^1_k$ is separated. $\mathbb{P}^1_k$ is covered by the two affine subsets $U_1 = \operatorname{Spec} k[x]$ and $U_2 = \operatorname{Spec} k[x^{-1}]$, which have affine intersection $\operatorname{Spec} k[x, x^{-1}]$. To conclude, we need only check that the map

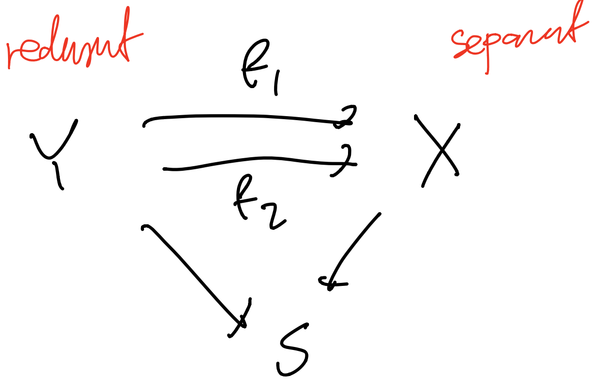
$$k[x] \otimes k[x^{-1}] \to k[x, x^{-1}]$$

is surjective, and it is. ☆

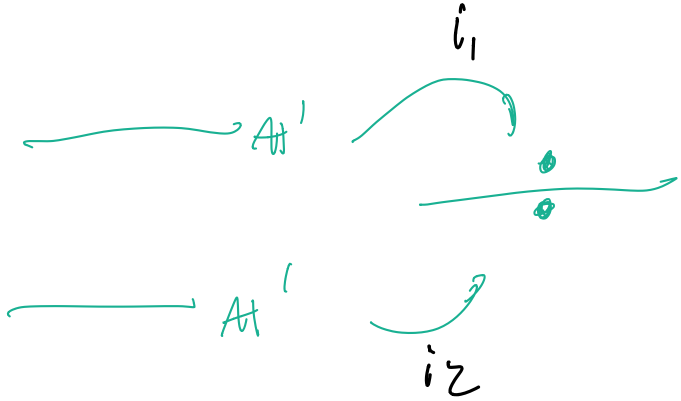


Example 8.13 Here is a non-separated scheme where two affine open sets have non-affine intersection. We glue two copies of the affine plane $\mathbb{A}^2_k$ together along the complement $U_{12} = \mathbb{A}^2_k - V(x,y)$ of the origin. If U_1 and U_2 denote the two open immersions of the affine plane, then $U_1 \cap U_2 = U_{12}$, but the open set U_{12} is not affine (see the example in Section 5.1 on page 94). ☆

Another useful property is that morphisms into separated schemes are determined on open dense sets, at least when the source is reduced:



Proposition 8.14 *Let X and Y be two schemes over S and $f_1, f_2 \colon Y \to X$ two morphisms over S. Assume that Y is a reduced scheme and X is separated over S. Moreover assume there is an open immersion $\iota \colon U \to Y$ with dense image such that $f_1 \circ \iota = f_2 \circ \iota$. Then $f_1 = f_2$.*



Example 8.15 The above proposition fails when X is not separated. For instance, if X is the affine line with two origins, then there are two morphisms $i_i : \mathbb{A}^1_k \to X$ for $i = 1, 2$ which agree on a dense open set, but they are not equal. ☆

Example 8.16 Likewise, if Y is non-reduced: Let $Y = \operatorname{Spec} k[x,y]/(y^2, xy)$ and consider the two maps $f_i : Y \to \operatorname{Spec} k[u]$ $i = 1, 2$ defined by $u \mapsto x$ and $u \mapsto x + y$ respectively. These agree over the distinguished open set $D(x)$, but they are different. ☆

Varieties vs schemes again

With the notion of separatedness, we can finally state the definition of a variety:

Definition 8.17 *A* variety X *is an integral, separated scheme of finite type over an algebraically closed field.*

This definition should be compared with the definition from Chapter 4. There we defined a variety to be a scheme in the image of the functor

$$\mathsf{Var}/k \to \mathsf{Sch}/k$$

which associates a k-variety V to a scheme V^s over k. As varieties satisfy the Hausdorff axiom, it is immediate that the corresponding scheme V^s is separated. Thus the two notions agree.

From now on a "variety" will always refer to a scheme satisfying Definition 8.17. Basically any theorem from the "classical setting" regarding varieties carry over to varieties in the new sense. This is justified by the following theorem:

Theorem 8.18 *The functor* $V \to V^s$ *is* fully faithful*, and gives and equivalence between the category of varieties* Var/k *and the subcategory of* Sch/k *of schemes satisfying Definition 8.17.*