Chapter 9

Projective schemes

$k = \mathbb{F}_2$

Let us consider the usual construction of complex projective space: As a topological space, $\mathbb{CP}^n$ is the quotient space

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - 0\right) / \mathbb{C}^*$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by scaling the coordinates. Of course the orbits of $\mathbb{C}^*$ in $\mathbb{C}^{n+1} \backslash \{0\}$ are just the lines through the origin, which is the traditional "variety-way" of thinking about $\mathbb{CP}^n$.

We can translate this into algebra as follows: if f is a function on $\mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}^*$ a complex number, we get a new function f^λ by defining $f^\lambda(x) = f(\lambda x)$, and this gives an action of $\mathbb{C}^*$ on the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$.

⇝ $R = \mathbb{C}[x_0, \ldots, x_n]$ decomposes into eigenspaces $R = \bigoplus_{d \geq 0} R_d$

where $\lambda \in \mathbb{C}^*$ acts on R_d by $\lambda^d \cdot f$.

⇝ the action gives the usual grading on R.

Leaving the realm of complex manifolds and entring the world of schemes, we want to take the quotient of $\mathbb{A}_{\mathbb{C}}^{n+1} - 0 = \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_n] - V(x_0, \ldots, x_n)$ by this action. We write $\mathbb{P}_{\mathbb{C}}^n$ for the corresponding quotient space equipped with the quotient topology. The notation $\mathbb{P}_{\mathbb{C}}^n$, rather than $\mathbb{CP}^n$, is used to emphasise

$\mathbb{P}^n_{\mathbb{C}}$ = " Spec $\mathbb{C}[x_0 \dots x_n] - 0 \,/\, \mathbb{C}^*$ "

Lettere å definere $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ der $U_i \simeq \mathbb{A}^n$.

Topologi definert ved en morfi Spec $\mathbb{C}[x_0 \dots x_n] - 0 \xrightarrow{\pi} \mathbb{P}^n$
og $V \subset \mathbb{P}^n$ åpen $\iff$ $\pi^{-1}(V)$ åpen.

We can try to put a scheme structure on $\mathbb{P}^n_{\mathbb{C}}$ by looking for reasonable open covers. Note that the open subsets of $\mathbb{P}^n_{\mathbb{C}}$ correspond to $\mathbb{C}^*$-invariant open subsets of $\mathbb{A}^{n+1}_{\mathbb{C}} - 0$. It is not too hard to see that $D(f) \subset \mathbb{A}^{n+1}_{\mathbb{C}}$ is $\mathbb{C}^*$-invariant if and only if f is a homogeneous polynomial. We write $D_+(f) \subset \mathbb{P}^n_{\mathbb{C}}$ for the open subset corresponding to $D(f) \subset \mathbb{A}^{n+1}_{\mathbb{C}} - 0$.

To define a structure sheaf on $\mathbb{P}^n_{\mathbb{C}}$ we must figure out what the spaces of sections $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(D_+(f))$ should be. While it is true that $D(f)$, being an affine scheme, has a structure sheaf, we have to take more care in deciding which sections to take, to make things compatible with the $\mathbb{C}^*$-action: a function on $D_+(f)$ should be a function on $D(f)$ that is invariant under the action of $\mathbb{C}^*$.

That is, we should have $g^\lambda = g$, which means precisely that g has degree zero. Thus we define

$$\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(D_+(f)) = \mathbb{C}[x_0, \ldots, x_n, f^{-1}]_0.$$

where the subscript means that we take the degree 0 part.

We can generalize the above for any affine $\mathbb{C}$-scheme with an action of $\mathbb{C}^*$. Such a scheme corresponds to a graded $\mathbb{C}$-algebra R. To make a reasonably good quotient, it is necessary remove the locus in $\operatorname{Spec} R$ that is fixed by $\mathbb{C}^*$, and it is not too hard to prove the following:

Lemma 9.1 *The fixed locus of $\mathbb{C}^*$ acting on $\operatorname{Spec} R$ is $V(R_+)$, where R_+ denotes the ideal generated by element of positive degree.*

We then proceed to consider the quotient space P of $\operatorname{Spec} R - V(R_+)$ by $\mathbb{C}^*$. Again, the $\mathbb{C}^*$-invariant distinguished open subsets in $\operatorname{Spec} R$ of the form $D(f)$ where f is homogeneous constitute a basis for the topology on $\operatorname{Spec} R - V(R_+)$, and these correspond to open subsets $D_+(f) \subset P = (\operatorname{Spec} R - V(R_+)) / \mathbb{C}^*$, which form a basis for the quotient topology. Finally, we define a $\mathscr{B}$-sheaf on P by setting $\mathcal{O}_P(D_+(f)) = \mathcal{O}_{\operatorname{Spec} R}(D(f))_0$, and check that we get a scheme P.

Beside of inducing a grading on R, the action of $\mathbb{C}^*$ plays very little role here. Realizing this, we can in fact build a scheme P from any graded ring R: We construct the topological space of P from the set of *homogeneous* prime ideals of R (with the induced Zariski topology), and define a structure sheaf on it by the formula like the one above. This is essentially the 'Proj'-construction.

9.2 *Basic remarks on graded rings*

A *graded ring* R is a ring with a decomposition

$$R = \bigoplus_{n \in \mathbb{N}_0} R_n = R_0 \oplus R_1 \oplus \cdots$$

as an abelian group such that $R_m \cdot R_n \subset R_{m+n}$ for each $m, n \geqslant 0$. Note that R_0 is a subring of R and that each of the R_n's is an R_0-module. The elements in R_n are said to be *homogeneous* of degree n, and one writes $\deg x = n$ when $x \in R_n$.

Every element $x \in R$ can be expressed unambiguisely as a sum $x = \sum_{n \in \mathbb{N}_0} x_n$ with $x_n \in R_n$. The non-zero terms in the sum are called the *homogeneous components* of x.

An R-module M is *graded* if it has a similar decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ as an abelian group such that $R_m \cdot M_n \subset M_{m+n}$ for all. A map of graded R-modules is an R-linear map $\phi : M \to N$ such that $\phi(M_n) \subset N_n$ for all $n \in \mathbb{Z}$. Note that contrary to what we required for maps between graded rings, degrees are preserved.

As usual, a non-zero element $x \in M$ is *homogeneous* of degree n if it lies in M_n. Just like ring elements, any member $x \in M$ may be expressed in a unique way as $x = \sum_{n \in \mathbb{N}_0} x_n$ with each x_n in M_n, and the non-zero terms are called the *homogeneous components* of x.

An ideal $\mathfrak{a} \subset R$ is *homogeneous* if the homogeneous components of each element in $\mathfrak{a}$ belongs to $\mathfrak{a}$. This is the case if and only if $\mathfrak{a}$ is generated by homogeneous elements. It is readily verified that intersections, sums and products of homogeneous ideals are homogeneous.

We will write R_+ for the sum $\bigoplus_{n>0} R_n$; this is naturally a homogeneous ideal of R, which we call the *irrelevant ideal*.

ex

$R = k[x_0, x_1]$ deg $x_i = 1$

$R_+ = (x_0, x_1)$

"Veronese underlying"

We let $R^{(d)}$ denote the subring of R given by $\bigoplus_{n \geqslant 0} R_{nd}$

Localization

Occasionally we shall meet graded rings having elements of negative degree; they are defined as above except that they decompose as

$$R = \bigoplus_{n \in \mathbb{Z}} R_n.$$

Some authors refer to these as $\mathbb{Z}$-graded rings. One way such rings appear is as localizations of graded rings. Indeed, if $T \subset R$ is a multiplicative system all whose elements are homogeneous, one may define a grading on $T^{-1}R$ by letting $\deg g/t = \deg g - \deg t$ for $t \in T$ and g a homogeneous element from R. In other words, one puts

$$(T^{-1}R)_n = \{ f/t \in T^{-1}R \mid f \in R_n, t \in T \text{ and } \deg f - \deg t = n \}.$$

Then, as is easily verified, the localized ring $T^{-1}R$ decomposes as the direct sum as $T^{-1}R = \bigoplus_{n\in\mathbb{Z}}(T^{-1}R)_n$, which makes it a $\mathbb{Z}$-graded ring. The same construction also works very well for graded modules, so that $T^{-1}M$ is a graded module whose homogeneous elements are of shape xt^{-1} with x homogeneous and $\deg xt^{-1} = \deg x - \deg t$.

One example of multiplicative sets of the graded sort, are the sets $T(\mathfrak{p})$ consisting of all homogeneous elements in R not lying in a given homogeneous prime ideal $\mathfrak{p}$. Another example is the set S of non-negative powers of a homogeneous element f.

$k[x_0, x_1]_{(x_0)}$ $\qquad \frac{x_1}{x_0}$

$k[x_0, x_1, x_2]_{((x_0, x_1))}$ $\qquad \frac{x_1}{x_2}$

Definition 9.2 *For a homogeneous prime ideal $\mathfrak{p} \subset R$ and a homogeneous element $f \in R$, we define for an R-module M*

i) $M_{(\mathfrak{p})} = (T(\mathfrak{p})^{-1}M)_0$; ← grad 0

ii) $M_{(f)} = (M_f)_0$; ←

where the subscript indicates the degree 0 *part.*

Example 9.3 For the polynomial ring $R = A[x_0, \ldots, x_n]$ with standard grading, the degree 0 part of R_{x_j} is generated by the monomials $x_0 x_j^{-1}, \ldots, x_j^{-1}$, so

$$R_{(x_j)} = A[x_0 x_j^{-1}, \ldots, x_n x_j^{-1}].$$

☆

ex $k[x_0, x_1]$ $\deg x_0 = 1$, $\deg x_1 = 2$

$k[x_0, x_1]_{(x_0)} \ni \frac{x_1}{x_0^2}$

$$R_+ = \bigoplus_{d>0} R_d$$

9.3 *The Proj construction*

Motivated by the discussion in the introduction, we make the following definition:

DEFINITION 9.4 *Let R be a graded ring. We denote by* Proj R *the set of homogeneous prime ideals of R that do not contain the irrelevant ideal R_+.*

$$\operatorname{Proj} R = \left\{ \mathfrak{p} \subset R \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ homogent primideal} \\ \mathfrak{p} \not\supset R_+ \end{array} \right\}$$

$$\subset \operatorname{Spec} R$$

One endows Proj R with a topology by setting, for a homogeneous ideal $\mathfrak{b}$,

$$V(\mathfrak{b}) = \{\, \mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{p} \supset \mathfrak{b} \,\},$$

and just like in the case of Spec R, these sets comply to the axioms for the closed sets of a topology, which is called the *Zariski topology* on Proj R. Indeed, the *i*) $V(\sum \mathfrak{b}_\mathfrak{i}) = \bigcap V(\mathfrak{b}_\mathfrak{i})$;

ii) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$;

iii) $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$,

the pertinent remark being that sums, products and radicals persist being homogeneous when the involved ideals are. Notice that this topology is nothing but the one induced from the inclusion $\operatorname{Proj} R \subset \operatorname{Spec} R$.

$$D_+(f) = \{ \mathfrak{p} \in \operatorname{Proj} R \mid f \notin \mathfrak{p} \}$$
$$= D(f) \cap \operatorname{Proj} R$$

As with the affine case, we define distinguished open sets. For $f \in R$ homogeneous of positive degree, we let $D_+(f)$ be the collection of homogeneous ideals (not containing R_+) that do not contain f, or in other words, $D_+(f) = D(f) \cap \operatorname{Proj} R$. These are open sets with respect to the Zariski topology on $\operatorname{Proj} R$ the complement of $D_+(f)$ equals the closed set $V(f)$.

$\rightsquigarrow$ $D_+(f)$ gir en basis for Zariski topologien på Proj R.

The relevance of the name the irrelevant ideal is that R_+ does not play any role when it comes to forming closed sets in $\operatorname{Proj} R$, neither do ideals whose radical equals R_+. This is made clear by the following lemma. Note that $V(R_+) = \varnothing$ by definition.

Lemma 9.5 *For any homogeneous ideal* $\mathfrak{a}$ *it holds that* $V(\mathfrak{a}) = V(\mathfrak{a} \cap R_+)$. *In fact, if* $\mathfrak{I}$ *is an ideal such that* $\sqrt{\mathfrak{I}} = R_+$, *it holds that* $V(\mathfrak{a}) = V(\mathfrak{a} \cap \mathfrak{I})$.

Proof: Since $V(R_+) = \emptyset$, condition *iii)* above implies that $V(\mathfrak{I}) = \emptyset$, and condition *ii)* then gives $V(\mathfrak{a} \cap \mathfrak{I}) = V(\mathfrak{a}) \cup V(\mathfrak{I}) = V(\mathfrak{a})$. ❑

The next result is important in understanding the local structure of $\operatorname{Proj} R$. In particular, it will be essential when defining the scheme structure on it.

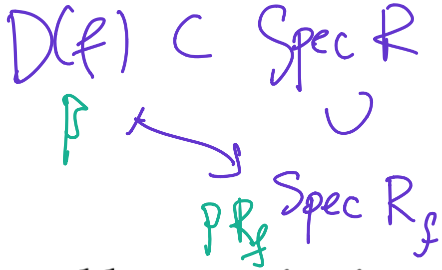
Proposition 9.6 *We have $D_+(f) \cap D_+(g) = D_+(fg)$. Also, the $D_+(f)$ form a basis for the topology on $\operatorname{Proj} R$ when f runs through the homogeneous elements of R of positive degree.*

$D_+(f) \cap D_+(g) = D_+(fg)$

$\mathfrak{p} \in RHS \Rightarrow \mathfrak{p} \not\ni f \cdot g \Leftrightarrow \mathfrak{p} \not\ni f$ og $\mathfrak{p} \not\ni g$

$\Leftrightarrow \mathfrak{p} \in LHS$

Proof: The first part is evident, by the definition of a prime ideal. We prove the second. Note that $V(\mathfrak{a})$ is the intersection of the $V((f))$'s for the homogeneous $f \in \mathfrak{a} \cap R_+$. Thus $\operatorname{Proj} R - V(\mathfrak{a})$ is the union of these $D_+(f)$. So every open set is a union of sets of the form $D_+(f)$. ❑



Dehomogenization and homogenization

In the affine case there is a canonical homeomorphism between $D(f)$ and $\operatorname{Spec} R_f$ which associates $\mathfrak{p}R_f$ with a prime $\mathfrak{p} \in D(f)$. In perfect analogy with this, associating the degree zero part of $\mathfrak{p}R_f$ with $\mathfrak{p} \in D_+(f)$ gives a homeomorphism between $D_+(f)$ and $\operatorname{Spec} R_{(f)}$.

Example 9.7 To illustrate this correspondence in a simple case which hopefully eases the digestion of the general case, let us consider the ring $R = k[x,y,z]$, and the distinguished open set $D_+(z)$. The monomials of degree zero in R_z are products of xz^{-1} and yz^{-1} so that $R_{(z)} = k[xz^{-1}, yz^{-1}]$. Consider a principal ideal $\mathfrak{a} = (f)$ in R generated by a homogeneous polynomial f of degree d. Because z is invertible in R_z and because of the identity

$$f(xz^{-1}, yz^{-1}, 1) = z^{-d} f(x,y,z),$$

the ideal $\mathfrak{a}R_z$ becomes $\mathfrak{a}R_z = (z^{-d}f)$, and since $z^{-d}f$ is of degree zero, it holds true that $(\mathfrak{a}R_z)_0 = \mathfrak{a}R_z \cap R_0 = (z^{-d}f)$. So when we pass to $R_{(z)}$, the generator f is replaced by the *dehomogenized* polynomial $z^{-d}f$.

There is also simple way of making a polynomial g in $k[xz^{-1}, yz^{-1}]$ homogeneous, one simply gives g a factor z^d with d being the degree of g. This will almost all the time be an inverse to the dehomogenization process; there is just one fallacy, any factor of f which is a power of z, disappears when f is dehomogenized, and there is no means of recovering it knowing only $z^{-d}f$. ☆

The general set up of the isomorphism $D_+(f) \simeq \operatorname{Spec} R_{(f)}$ follows the pattern in the example, basically one dehomogenizes and homogenizes generators, but expressed in a necessarily general formalism.

$M_f \longrightarrow M_g \quad \rightsquigarrow \quad M_{(f)} \to M_{(g)}$

Proposition 9.8 *Let $f \in R$ be homogeneous of degree d. There is a canonical homeomorphism $\phi : D_+(f) \to \operatorname{Spec} R_{(f)}$ given by*

$$\phi(\mathfrak{p}) = \mathfrak{p}R_f \cap R_{(f)},$$

that sends homogeneous prime ideals of R not containing f into primes of $R_{(f)}$. Moreover,

i) For any homogeneous $g \in R$ such that $D_+(g) \subset D_+(f)$, letting $u = g^d f^{-\deg g} \in R_{(f)}$, we have $\phi(D_+(g)) = D(u)$;

ii) For any graded R-module M, there is a canonical homomorphism $M_{(f)} \to M_{(g)}$ which induces an isomorphism $(M_{(f)})_u \simeq M_{(g)}$; $\rightsquigarrow$ B-knippe.

iii) If $\mathfrak{a} \subset R$ is a homogeneous ideal, then $\phi(V(\mathfrak{a}) \cap D_+(f)) = V(\mathfrak{a}R_f \cap R_{(f)})$.

$\rightsquigarrow$ se heftet for beviset.

Proj R *as a scheme*

We shall now make $X = \operatorname{Proj} R$ into a locally ringed space. Let $\mathscr{B}$ be the base of Proj R made up by the distinguished open subsets.

$$\mathcal{B} = \{ D_+(f) \mid f \in R \text{ homogen} \}$$

For each $D_+(f)$ we define

$$\mathcal{O}(D_+(f)) = R_{(f)}.$$

grad 0 biten i R_f.

$\frac{a}{f^n}$

$\operatorname{Spec} R_{(f)} = D_+(f)$

$\deg a = n \cdot \deg f$

$\Rightarrow \mathcal{O}_{D_+(f)}(D(u)) = (R_{(f)})_u$

$\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$

$$\rightsquigarrow \quad \mathcal{O}|_{D_+(f)} = \mathcal{O}_{D_+(f)}$$ (som definert på affine skjemaer)

The previous proposition shows that this gives a well-defined $\mathscr{B}$-presheaf $\mathcal{O}$ of rings, and using the homeomorphism ϕ from $D_+(f)$ to Spec $R_{(f)}$, we see that it actually is a $\mathscr{B}$-sheaf. (Alternatively, we could modify the proof for the case of Spec to see this directly). We will denote the unique sheaf extension by $\mathcal{O}_X$.

$$\mathcal{O}(D_+(f)) = R_{(f)}$$

stilkene er lokale ringer !

It follows that X is has the structure of a ringed space. This is in fact a *locally* ringed space, because the stalk $\mathcal{O}_{X,x}$ is just $R_{(\mathfrak{p}_x)}$, which is a local ring. Indeed, the unique maximal ideal is generated by $\mathfrak{p}$. Moreover, the previous discussion has shown that the basic open sets $D_+(f)$ are each isomorphic as locally ringed spaces to $\operatorname{Spec} R_{(f)}$, which are affine schemes, and so $\operatorname{Proj} R$ is a scheme.

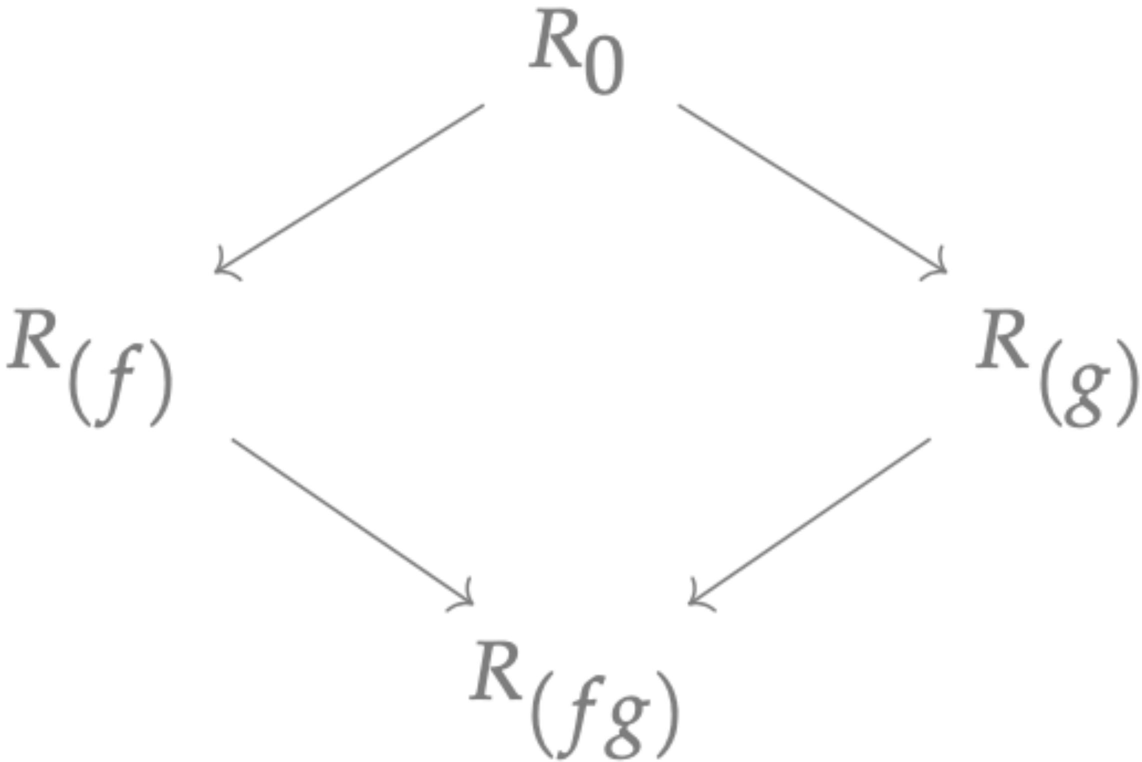
Definition 9.9 *For a graded ring R, we call the scheme* $(\operatorname{Proj} R, \mathcal{O}_{\operatorname{Proj} R})$ *the* projective spectrum *of R.*

In fact, the projective spectrum $\operatorname{Proj} R$ is naturally a scheme over $\operatorname{Spec} R_0$: the homomorphisms $R_0 \to R_{(f)}$ induce maps $\operatorname{Spec} R_{(f)} \to \operatorname{Spec} R_0$, and these glue together to a morphism

$$\operatorname{Proj} R \to \operatorname{Spec} R_0,$$

ex $R = k[x_0, \dots, x_n]$

$\rightsquigarrow \mathbb{P}^n_k = \operatorname{Proj} R \longrightarrow \operatorname{Spec} k.$


R_0
$R_{(f)}$
$R_{(g)}$
$R_{(fg)}$

Moreover, if R is a finitely generated over R_0, the spectrum $\operatorname{Proj} R$ is of finite type over $\operatorname{Spec} R_0$. This follows by looking at the distinguished open sets $D_+(f)$ – each ring $R_{(f)}$ is finitely generated as an R_0-algebra if R is.

Definition 9.10 *We define the* projective n-space *to be the scheme*

$$\mathbb{P}^n = \operatorname{Proj} \mathbb{Z}[x_0, \dots, x_n].$$

More generally, for a ring A, the projective n-space over A *is the scheme*

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n].$$

S skjema $\rightsquigarrow$ $\mathbb{P}^n_S$ = relativt projektivt rom.

$$R_0 = A \hookrightarrow R_{(f)}$$

$$\operatorname{Spec} R_{(f)} \to \operatorname{Spec} A$$

$$\rightsquigarrow \operatorname{Proj} R \to \operatorname{Spec} A$$

Examples

9.11 Let A be a ring and let $R = A[t]$ with the grading given by $\deg t = 1$ and $\deg a = 0$ for all $a \in A$. Then the structure map gives an isomorphism $\operatorname{Proj} R \simeq \operatorname{Spec} A$.

$\therefore$ Proj R kan være affin.

$$\mathfrak{p} \in \operatorname{Proj} R \rightsquigarrow \mathfrak{p} \cap R_0 \in \operatorname{Spec} A$$

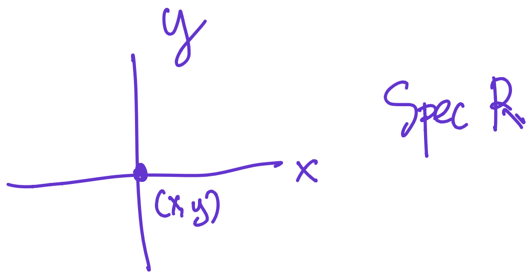
9.12 *(The projective line $\mathbb{P}^1_A$ once more)* Let us study the case of a polynomial ring in $R = A[s,t]$ where s and t have degree one. The scheme $X = \operatorname{Proj} R$ coincides with $\mathbb{P}^1_A$ as defined in Chapter 5 (in Section 5.1 on page 94); indeed, we shall see that it is glued together from affine schemes in precisely the same manner as is $\mathbb{P}^1_A$. Note that X is covered by $D_+(s)$ and $D_+(t)$ (since s and t generate the irrelevant ideal). Write for simplicity $U = D_+(s) \simeq \operatorname{Spec} R_{(s)}$ and $V = D_+(t) \simeq \operatorname{Spec} R_{(t)}$. It holds true that X is glued together from U, V along $U \cap V = D_+(st) \simeq \operatorname{Spec} R_{(st)}$.

Note first that the degree zero part $R_{(s)}$ of $R_s \simeq A[s, s^{-1}, t]$ equals $A[s^{-1}t]$, and by symmetry we have $R_{(t)} = A[st^{-1}]$. The intersection $D_+(st)$ is the degree zero part of R_{st} which is given as $R_{(st)} = A[s^{-1}t, st^{-1}]$. In other words, if we write $u = s^{-1}t$, it holds true that $R_{(s)} = A[u]$, $R_{(t)} = A[u^{-1}]$ and that $R_{(st)} = A[u, u^{-1}] = A[u]_u$. Hence $U \simeq \operatorname{Spec} A[u] = \mathbb{A}^1_A$ and $V \simeq \operatorname{Spec} A[u^{-1}] \simeq \mathbb{A}^1_A$ are glued together along $\operatorname{Spec} R_{(st)}$, and this is exactly the glueing scheme used to construct $\mathbb{P}^1_A$ in Section 5.1.

9.13 *(Projective n-space)* The case when $R = k[x_0, \ldots, x_n]$ is a polynomial ring over a field k is the most interesting. In this case $\mathbb{P}^n_k$ is a scheme whose closed k-points $\mathbb{P}^n(k)$ coincides with the *variety* of projective n-space.

Since $\mathbb{P}^n_k$ is covered by $n+1$ copies of $\mathbb{A}^n_k$, $\mathbb{P}^n_k$ is integral of dimension n. We also have $k(\mathbb{P}^n_k) = k(\mathbb{A}^n_k) = k(X_1, \ldots, X_n)$. More intrinsically, we may also write

$$k(\mathbb{P}^n) = \left\{ \frac{g(x_0, \ldots, x_n)}{h(x_0. \ldots, x_n)} \,\middle|\, g, h \text{ homogeneous of the same degree} \right\}$$



9.14 Let $R = k[x,y]/(xy)$. Spec R is the union of the x- and y-axes. So Spec R – $V(x,y)$ is the union of the axes with the origin excluded. On the other hand,

$$\left(k[x]_x\right)_0 = \left(k[x, x^{-1}]\right)_0 = k$$

Proj R consists of just two points: Proj R is obtained by gluing Spec$(R_{(x)})$ and Spec$(R_{(y)})$ together. Now,

$$R_{(x)} = k[x,y]_{(x)}/xy = k[x,y]_{(x)}/y = k[x]_{(x)} = k,$$

and the corresponding chart of Proj R is just Spec k. Similarly, the other chart Spec$(R_{(y)})$ also equals Spec k. We have $R_{(xy)} = 0$, so the overlap is empty, and it ensues that Proj R consists of the two points.

Proposition 9.16 (Properties of Proj R) *Let R be a graded ring.*

i) $\operatorname{Proj} R$ *is separated.*

ii) *If R is noetherian, then* $\operatorname{Proj} R$ *is noetherian.*

iii) *If R is finitely generated over R_0, then* $\operatorname{Proj} R$ *is of finite type over* $\operatorname{Spec} R_0$.

iv) *If R is an integral domain, then* $\operatorname{Proj} R$ *is integral.*

$D_+(f) = \operatorname{Spec} R_{(f)}$ integral

Proof: We use the fact that X is covered by the affine open sets $D_+(f)$ where f runs over the elements of R^+. These sets are clearly affine, and so is their intersection: $D_+(f) \cap D_+(g) = D_{(fg)}$. Thus to prove that $\operatorname{Proj} R$ is separated,

- U_i affin overdeling
- $U_i \cap U_j$ affin $\forall i,j$
- $\Gamma(U_i, \mathcal{O}) \otimes \Gamma(U_j, \mathcal{O}) \twoheadrightarrow \Gamma(U_{ij}, \mathcal{O})$ $\forall i,j$

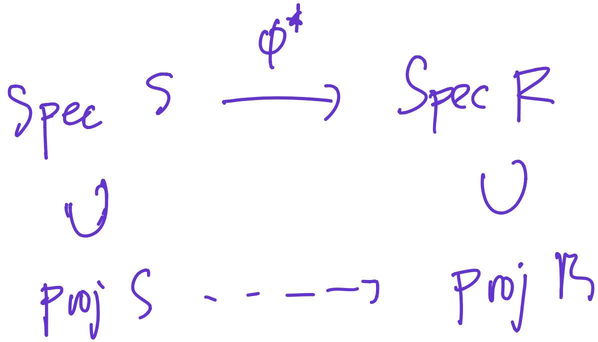
$\Rightarrow$ X separeret

$R_{(f)} \otimes R_{(g)} \to R_{(fg)}$

$R_{(fg)}$

we need only check condition *ii)* above, namely that $R_{(f)} \otimes R_{(g)} \to R_{+}(fg)$ is surjective for any $f, g \in R^{+}$, but this is straightforward.

The remaining properties are properties which can be checked on an affine covering. In our case $\operatorname{Proj} R$ is covered by the affines $\operatorname{Spec} R_{(f)}$ which are noetherian (resp. of finite type, integral) provided R is noetherian (resp. finitely generated, an integral domain). ❑



9.4 *Functoriality*

Unlike the case of affine schemes, a graded ring homomorphism $\phi : R \to S$ does not induce a morphism between the projective spectra $\operatorname{Proj} S$ and $\operatorname{Proj} R$. The reason is that some primes in S may pullback to R to contain the irrelevant ideal R_+. However, as we will see shortly, this is the only obstruction to defining a morphism.

Given a homomorphism $\phi : R \to S$, we define the set $G(\phi) \subset \operatorname{Proj} S$ to be the set of homogeneous prime ideals $\mathfrak{p}$ in S that do not contain $\phi(R_+)$, in particular those prime ideals have their inverse images $\phi^{-1}(\mathfrak{p})$ in $\operatorname{Proj} R$. The assignment $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ then sets up a map

$$F : G(\phi) \to \operatorname{Proj} R.$$

$\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$

[illegible] R_+.

The set $G(\phi)$ is an open subset of $\operatorname{Proj} S$; indeed, let $\mathfrak{p} \subset S$ be a homogeneous prime ideal in $G(\phi)$. Then $\mathfrak{p}$ does not contain $\phi(R_+)$, so (assuming $\phi(R_+) \neq 0$) there exists an $r \in R_+$ such that $s = \phi(r) \notin \mathfrak{p}$, and we may clearly assume that s is homogeneous. It the holds that $\mathfrak{p} \in D_+(s)$, but also $D_+(s) \subset G(\phi)$ since each $\mathfrak{q} \in D_+(s)$ does not contain s. Hence $G(\phi)$ is open. That $\phi(R_+) = 0$ implies that $G(\mathfrak{p}) = \varnothing$, and $G(\phi)$ is open in that case also.

Being an open subset $G(\phi)$ has the canonical induced scheme structure as an open subscheme of $\operatorname{Proj} S$, and giving it that structure, we have:

Proposition 9.18 *Let $\phi : R \to S$ be a homomorphism of graded rings. Then the map $F : G(\phi) \to \operatorname{Proj} R$ is a morphism of schemes.*

Proof: First of all, the map F is continuous because the Zariski topologies on $\operatorname{Proj} R$ and $\operatorname{Proj} S$ are induced from those of $\operatorname{Spec} S$ and $\operatorname{Spec} R$, and because F is the restriction of the map between the two Spec's induced by ϕ. Or more explicitly, the inverse image $F^{-1}(D_+(f))$ equals $G(\phi) \cap D_+(\phi(f))$, which is open.

Write $X = G(\phi)$ and $Y = \operatorname{Proj} R$. The rest of the job is to define the map F on the level of sheaves, *i.e.* we desire a map

$$F^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X.$$

As usual, it suffices to define it on the basis of distinguished open subsets. To define it on $D_+(f) \subset \operatorname{Proj} R$ we rely on the isomorphism between $D_+(f)$ and $\operatorname{Spec} R_{(f)}$ from Proposition 9.8. Of course, only opens $D_+(f)$ so that $F^{-1}(D_+(f))$ are non empty matter; then $\phi(f) \notin R_+$, and then the localization of ϕ induces a map $R_{(f)} \to S_{\phi(f)}$. Moreover, since $F^{-1}(D_+(f))$ is open in $D_+(\phi(f)) = \operatorname{Spec} S_{(\phi(f))}$, we get the desired map

$$\mathcal{O}_Y(D_+(f)) = R_{(f)} \to \mathcal{O}_X\big(F^{-1}(D_+(f))\big)$$

by restriction of $\Gamma(D_+(f), \mathcal{O}_Y) \to \Gamma(D_+(\phi(f)), \mathcal{O}_{\operatorname{Proj} S})$. ❑

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^1 \qquad \mathbb{A}^3 \longrightarrow \mathbb{A}^2$$

$$(x_0, x_1, x_2) \longmapsto (x_0, x_1)$$

Example 9.19 To see why restriction to the open set $G(\phi)$ is necessary, we consider the case where $R = k[x_0, x_1]$, $S = k[x_0, x_1, x_2]$ and ϕ is the inclusion map. Note that the prime ideal $\mathfrak{a} = (x_0, x_1)$ defines an element in $\operatorname{Proj} S$, but its restriction to R is the whole irrelevant ideal of R. In fact, $G(\phi) = \operatorname{Proj} S - V(\mathfrak{a})$, and the map

$$\psi : \mathbb{P}^2_k - V(\mathfrak{a}) \to \mathbb{P}^1_k$$

is nothing but the projection from the point $(0 : 0 : 1)$ which sends a point with homogeneous coordinates $(x_0 : x_1 : x_2)$ to the one with coordinates $(x_0 : x_1)$. It is a good exercise to prove that there can be no morphisms $\mathbb{P}^m_k \to \mathbb{P}^n_k$ for $m > n$ in general. (See Section 16). ☆

Closed immersions

The primary example of the above construction is considering the graded quotient homomorphism $\phi : R \to R/\mathfrak{a}$, where $\mathfrak{a} \subset R$ is a homogeneous ideal. In this case $\phi(R_+) = (R/\mathfrak{a})_+$ so $G(\phi) = \operatorname{Proj}(R/\mathfrak{a})$, and the corresponding map ψ is defined everywhere; that is, we get a map

$$\psi : \operatorname{Proj}(R/\mathfrak{a}) \to \operatorname{Proj} R.$$

bildet = $V(\mathfrak{a})$

$$\operatorname{Proj} R/\mathfrak{a} \dashrightarrow \operatorname{Proj} R \qquad R_{(f)} \to \frac{R_{(f)}}{\mathfrak{a}}$$

$$\operatorname{Spec} (R/\mathfrak{a})_{(f)} \to \operatorname{Spec} R_{(f)}$$

$$D_+(\bar{f}) \longrightarrow D_+(f) \cap V(\mathfrak{a})$$

We claim that this is a closed immersion. As usual, homogeneous primes in $R/\mathfrak{a}$ not containing R_+ pull back to homogeneous primes in R containing $\mathfrak{a}$ but not R_+. It follows that ψ is injective with image the closed subset $V(\mathfrak{a})$ in Proj R. Finally, $\psi^\#$ is surjective on stalks (where it is just the map $R_{(\mathfrak{p})} \to (R/\mathfrak{a})_{(\mathfrak{p})}$), and so ψ is a closed immersion. We will see later that there is a converse to this statement, under some mild assumptions on R.

Example 9.20 The most simple conceivable closed immersion is that of a closed point in $\mathbb{P}^n_k$. At least if k is algebraically closed, such points are given by their *homogeneous coordinates* $a = (a_0 : \cdots : a_n)$, the maximal ideal corresponding is generated by the minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ a_0 & a_1 & \dots & a_n \end{pmatrix}. \tag{9.2}$$

~~$(x_1 - a_1, \dots, x_n - a_n)$~~ n

$(x_0 a_1 - x_1 a_0, \dots, x_{n-1} a_n - x_n a_{n-1})$ $\binom{n+1}{2}$

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$$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ a_0 & a_1 & \dots & a_n \end{pmatrix}. \tag{9.2}$$

Indeed, the vanishing of those minors describes vectors in k^{n+1} dependent on $(a_0, \dots, a_n)$; or in other words, points lying on the line through $(a_0, \dots, a_n)$.

There is an analogue of this for projective spaces $\mathbb{P}^n_A$ over an arbitrary ring A that to an n-tuple $a = (a_0, \ldots, a_n)$ of elements from A gives an A-point of $\mathbb{P}^n_A$; that is, a section of the structure map $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$. The appropriate necessary condition on the a_i's that generalize the condition familiar from the theory of varieties that not all a_i be zero, is that the a_i's generate the unit ideal in A. And two such tuples give the same section if and only if they are proportional by a unit from A.

Let $\mathfrak{a}$ be the ideal in $A[x_0, \ldots, x_n]$ generated by the minors of the matrix (9.2); in other words

$$\mathfrak{a} = (a_i x_j - a_j x_i \,|0 \leqslant i, j \leqslant n).$$

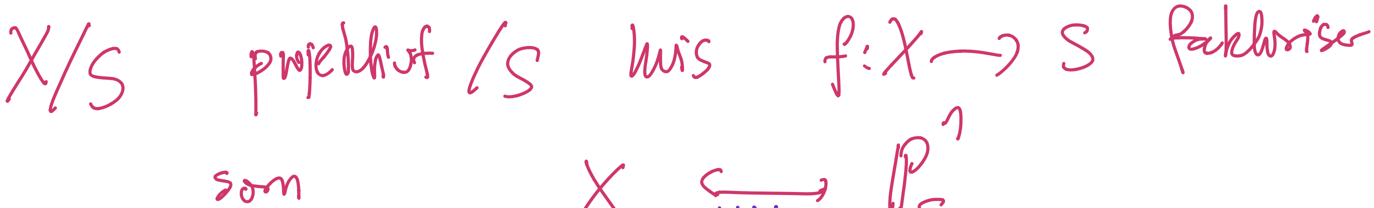
We claim that π induces an isomorphism between $V(\mathfrak{a})$ and $\operatorname{Spec} A$; its inverse will then be a closed embedding $\iota_a \colon \operatorname{Spec} A \to \mathbb{P}^n_A$. The open distinguished sets $D(a_i)$ cover $\operatorname{Spec} A$, and it will suffice to see that the restriction $\pi|_{\pi^{-1}(D(a_i))} \colon V(\mathfrak{a}) \cap \pi^{-1}(D(a_i)) \to D(a_i)$ is an isomorphism for each i. So replacing $\operatorname{Spec} A$ by $D(a_i)$, we may well assume that one of the a_i's, say a_0, is invertible. Since $a_0 x_i - a_i x_0$ belongs to $\mathfrak{a}$, we deduce that $x_i - a_i a_0^{-1} x_0 \in \mathfrak{a}$, and hence $A[x_0, \ldots, x_n]/\mathfrak{a} = A[x_0]$. By Example 9.11, it follows that the structure map restricts to an isomorphism on $V(\mathfrak{a})$. Clearly a simultaneous scaling does not change $a_i a_0^{-1}$, and if $a_i a_0^{-1} = a_i' a_0'^{-1}$, it holds that $a_i' = a_0' a_0^{-1} a_i$.

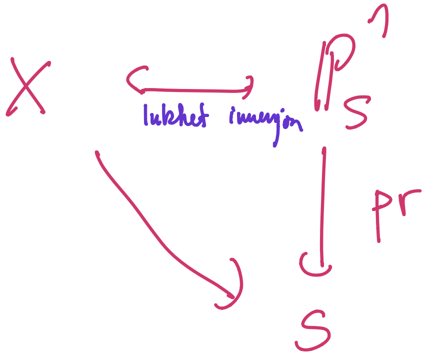
It is not true in general that all maps $\operatorname{Spec} A$ to $\mathbb{P}^n$ are of the "homogeneous coordinate form" $(a_1 : \cdots : a_n)$, but if A is local (e.g., a field) it holds true.

Lemma 9.21 *Assume that A is a local ring. Then every section $\operatorname{Spec} A \to \mathbb{P}^n_A$ of the structure map is given by $(a_1 : \cdots : a_n)$ where at least one a_i is a unit. Another such tuple $(a'_1 : \cdots : a'_n)$ gives the same map if and only if $a'_i = \alpha a_i$ for a unit $\alpha \in A$.*

One must remember that the lemma is relative to a *fixed* sequence of variables $x_0, \dots, x_n$.

PROOF: Assume that a morphism $f\colon \operatorname{Spec} A \to \mathbb{P}^1_A$ given. Then the image of the closed point lies in $D_+(x_\nu)$ for some ν and f factors through $D_+(x_\nu)$. This means that $f^\sharp$ is a map $k[x_\nu x_\nu^{-1}, \dots, x_n x_\nu^{-1}]$, the image of $a_i = f^\sharp(x_i x_\nu^{-1})$ are elements in A an $(a_0 : \cdots : 1 : \cdots : a_n)$ are appropriate homogeneous coordinates giving the map f (where the 1 is in the ν-th slot). ❑

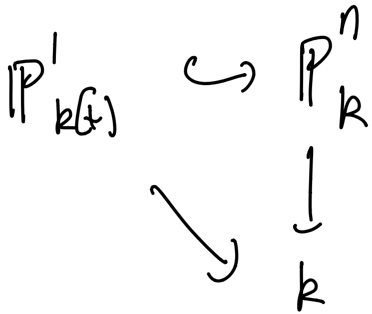




9.5 *Projective schemes*

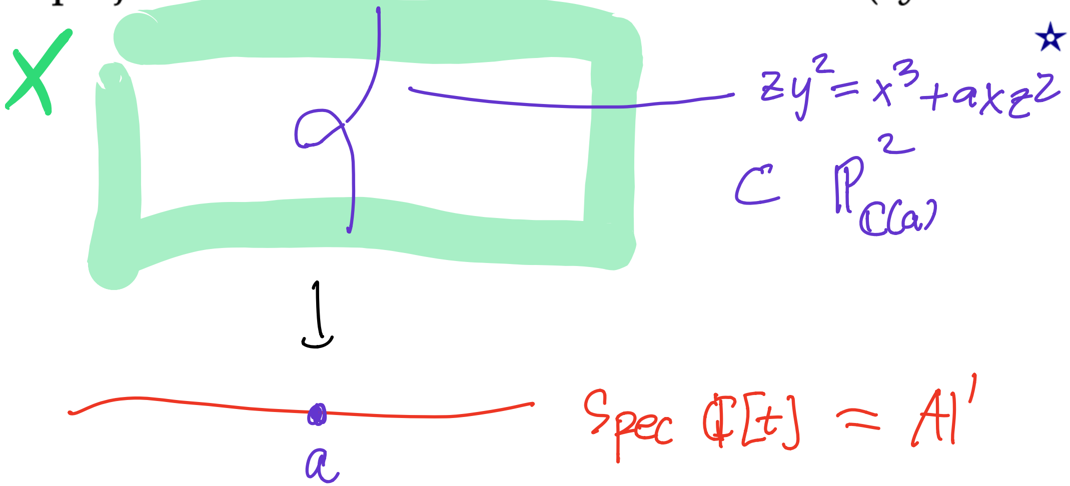
Let S be a scheme and let X be a scheme over S. We say call X is *projective* over S (or that the structure morphism $f : X \to S$ is projective) if $f : X \to S$ factors as $f = \pi \circ i$ where $i : X \to \mathbb{P}^n_S$ is a closed immersion and $\pi : \mathbb{P}^n_S \to S$ is the projection. X is *quasi-projective* over S if $X \to S$ factors via an open immersion $X \to \overline{X}$ and a projective S-mophism $\overline{X} \to S$.[1]

The primary examples is of course $X = \mathbb{P}^n_A \to \operatorname{Spec} A$ for a ring A. More generally, if $X = \operatorname{Proj} R$ where R is a graded R_0-algebra generated in degree 1 and $S = \operatorname{Spec} R_0$, then X is projective over S. In this case, we can define the projective immersion i by taking a surjection $R_0[x_0, \ldots, x_n] \to R$, which upon taking Proj, gives a closed immersion $X \to \mathbb{P}^n_{R_0}$.



Note that projectivity is a relative notion: It is the morphism $X \to S$ which is projective, not X itself. For instance, $\mathbb{P}^1_{k[t]}$ is projective over $\operatorname{Spec} k[t]$, but it is not over $\operatorname{Spec} k$. Still, if we are working in the category of schemes over, say, a field k or $\mathbb{Z}$, we still refer to a scheme X being 'projective' if it is projective over the base scheme.

Example 9.22 For $A = \mathbb{C}[t]$, the scheme $X = \operatorname{Proj} A[x,y,z]/(zy^2 - x^3 - txz^2)$ is projective over $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} A$. The preimage of $X \to \mathbb{A}^1_{\mathbb{C}}$ over any closed point $a \in \mathbb{A}^1_{\mathbb{C}}$ is an integral projective subscheme of dimension one: $V(zy^2 - x^3 - axz^2) \subset \mathbb{P}^2_{\mathbb{C}}$. ☆



$$R^{(d)} = \bigoplus_{k \geq 0} R_{dk} \to R$$

9.6 *The Veronese embedding*

Let R be a graded ring and let d be a positive integer. The inclusion $\phi : R^{(d)} \to R$ induces a morphism

$$v_d : \operatorname{Proj} R \to \operatorname{Proj} R^{(d)}$$

Indeed, in this case $G(\phi) = \operatorname{Proj} R$, since any prime $\mathfrak{p}$ such that $\mathfrak{p} \supset R_+ \cap R^{(d)}$ must also contain all of R_+ – if $r \in R_+$, note that $r^d \in R_+ \cap R^{(d)}$ and so $r \in \mathfrak{p}$ as well! This map is called the *Veronese embedding*, or *d-uple embedding* of X.

$\frac{k[u,v,w]}{(v^2 - uw)} = k[x^2, xy, z^2] \subset k[x,y]$

Proposition 9.23 *The Veronese embedding v_d is an isomorphism.*

Proof: There are many things to check here, so we will sketch the proof, and leave the remaining verifications for the reader.

v_d is injective: If $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} R$ such that $\mathfrak{p} \cap R^{(d)} = \mathfrak{q} \cap R^{(d)}$. Then for a homogeneous element $x \in R$ we have

$$x \in \mathfrak{p} \Leftrightarrow x^d \in \mathfrak{p} \Leftrightarrow x^d \in \mathfrak{q} \Leftrightarrow x \in \mathfrak{q}$$

and hence so $\mathfrak{p} = \mathfrak{q}$. To show that v_d is surjective, let $\mathfrak{q} \in \operatorname{Proj} R^{(d)}$, and define the homogeneous ideal in R by

$$\mathfrak{p} = \bigoplus_{n=0}^{\infty} \left\{ x \in R_n | x^d \in \mathfrak{q} \right\}.$$

It is not too hard to check that $\mathfrak{p}$ is prime, and that $\mathfrak{p} \cap R^{(d)} = \mathfrak{q}$, so v_d is bijective.

The proof then proceeds to show that the maps v_d and $\mathfrak{q} \mapsto \mathfrak{p}$ are closed, so that v_d is a homeomorphism. Then one checks directly that v_d induces an isomorphism when restricted to the open affines $D_+(f)$ as well, so we get an isomorphism on the level of schemes as well. ❑

Remark on rings generated in degree one We will frequently assume that the ring R is *generated in degree one*, that is, R is generated as an R_0-algebra by R_1. The reason for this will become clear in the next section. Intuitively, it is because we want $\operatorname{Proj} R$ to be covered by the 'affine coordinate charts' $D_+(x)$ where x should have degree 1.

We remark that this assumption is in fact not not too restrictive: Any projective spectrum of a finitely generated ring is isomorphic to the Proj of a ring generated in degree 1. This is because of the basic algebraic fact that if R is finitely generated, then some subring $R^{(d)}$ will have all of its generators in one degree, and since $\operatorname{Proj} R^{(d)} \simeq \operatorname{Proj} R$, we don't change the Proj by replacing R with $R^{(d)}$.

Example 9.24 *(The weigthed projective space $\mathbb{P}(p,q)$)* Let k be a field and p and q two relatively prime natural numbers and let $n = pq$. Consider the polynomial ring $R = k[x,y]$, but endow it with the non-standard grading with x having degree p and y degree q. We claim that $\operatorname{Proj} R \simeq \mathbb{P}^1_k$, or more specifically that $R^{(n)}$ is isomorphic to the polynomial ring $A = k[u,v]$ graded in the non-standard but innocuous way that $\deg u = \deg v = n$. Clearly $\operatorname{Proj} A \simeq \mathbb{P}^1_k$.

Observe that a homogeneous element in $R^{(n)}$ is a linear combination of monomials $x^\alpha y^\beta$ with $p\alpha + q\beta = \gamma n$; hence q divides α and p divides β and so $\alpha' + \beta' = \gamma$ with $\alpha = q\alpha'$ and $\beta = p\beta'$. There is a homomorphism of graded k-algebras $A = k[u, v] \to R^{(n)}$ that sends $u \to x^q$ and $v \to y^p$. This is clearly injective, so to see it is an isomorphism, it suffices to check it is surjective on each homogeneous component: now $(R^{(n)})_{dn}$ has a basis consisting of the monomials $x^{q\alpha'} y^{p\beta'}$ with $\alpha' + \beta' = d$; and for the same α' and β' the monomials $u^{\alpha'} v^{\beta'}$ form a basis for A_d. ☆

Example 9.25 *(The weighted projective space $\mathbb{P}(1,1,p)$)* Another example along the same lines. Again we begin with a polynomial ring $R = k[x,y,z]$ endowed with a slightly exotic grading; we put $\deg x = \deg y = 1$ and $\deg z = p$ for some natural number p. Then $\operatorname{Proj} k[x,y,z]$ is a so-called *weighted projective space* and one often sees it denoted by $\mathbb{P}(1,1,p)$.

The scheme $X = \operatorname{Proj} R$ has a covering of the three open affines $D_+(x)$, $D_+(y)$ and $D_+(z)$. Both $D_+(x)$ and $D_+(y)$ are isomorphic to $\mathbb{A}^2_k$; it is a straightforward exercise to verify that $R_{(x)} = k[yx^{-1}, zx^{-p}]$ and $R_{(y)} = k[xy^{-1}, zy^{-p}]$, and that these are polynomial rings. However the third distinguished open affine $D_+(z)$ is not isomorphic to $\mathbb{A}^2_k$. In fact, it has a singularity! Clearly $x^{p-i}y^iz^{-1}$, for $0 \leqslant i \leqslant p$, are homogeneous elements of degree zero in $R_{(z)}$, and it is almost trivial that they generate $R_{(z)}$, so that $R_{(z)} = k[x^pz^{-1}, \ldots, y^pz^{-1}]$. One recognizes this ring as an isomorphic copy of the p^{th} Veronese ring $A^{(p)}$ of the polynomial ring $A = k[u, v]$. And anticipating parts of the story, this is the cone over a so-called *projective normal curve* of degree p, whose apex is a singular point. ☆

Example 9.26 *(The Blow-up as a Proj)* Consider the ring $A = k[x, y]$ and the ideal $I = (x, y)$. We can form a new graded ring by introducing a new formal variable t and setting

$$R = \bigoplus_{k \geqslant 0} I^k t^k$$

where $I^0 = A$. In R, the new variable t has degree 1, and the other variables x and y have degree 0. One may think about R as the subring of $A[t]$ of polynomials shaped like $\sum_\nu a_v t^\nu$ where the coefficient a_ν belongs to I^ν.

The map $\mathfrak{p} \mapsto \mathfrak{p} \cap A$, induces a morphism

$$\pi : \operatorname{Proj} R \to \operatorname{Spec} A = \mathbb{A}^2_k$$

The irrelevant ideal R_+ is generated by xt and yt so that $\operatorname{Proj} R$ is glued together by the two open affine subschemes $\operatorname{Spec} R_{(xt)}$ and $\operatorname{Spec} R_{(yt)}$. These are both isomorphic to $\mathbb{A}^2_k$. Note that there is a map of graded rings

$$\begin{aligned} \phi : A[u,v] &\to R \\ u &\mapsto xt \\ u &\mapsto yt \end{aligned}$$

This is clearly surjective, since I is generated by x and y. Note also that the kernel contains the element $xv - yu$. In fact, by Exercise 9.12 below, we have

Lemma 9.27 $R \simeq A[u,v]/(xv - yu)$.

From this description we see that $\operatorname{Proj} R$ is covered by the two distinguished open sets $D_+(u) = \operatorname{Spec} R_{(v)}$ and $D_+(v) = \operatorname{Spec} R_{(v)}$. Here

$$R_{(u)} \simeq (A[u,v]_u/(xv - yu))_0 = k[x, vu^{-1}]$$

and

$$R_{(v)} \simeq (A[u,v]_v/(xv - yu))_0 = k[y, uv^{-1}].$$

These are glued along $R_{(uv)} \simeq (A[u,v]_{uv}/(xv - yu))_0$, and one finds

$$(A[u,v]_{uv}/(xv - yu))_0 = k[x, y, uv^{-1}, vu^{-1}]/(x \cdot vu^{-1} - y) \simeq k[x, uv^{-1}, vu^{-1}]$$

In particular, we see that $\operatorname{Proj} R$ coincides with the previous blow-up description.

☆