✳ 1.1 Let X be the set with two elements with the discrete topology. Find a presheaf on X which is not a sheaf.

$X = \{p, q\}$ ⇝ A constant sheaf works.

(gluing fails)

Here is an example which fails locality:

$\mathcal{F}(X) = \mathbb{Z}^3$

$\mathcal{F}(\{p\}) = \mathbb{Z}$

$\mathcal{F}(\{q\}) = \mathbb{Z}$

$\mathcal{F}(X) \to \mathcal{F}(\{p\})$ $\quad \mathbb{Z}^3 \xrightarrow{pr_1} \mathbb{Z}$

$\mathcal{F}(X) \to \mathcal{F}(\{q\})$ $\quad \mathbb{Z}^3 \xrightarrow{pr_2} \mathbb{Z}$

$$\begin{array}{ccccccc} & \mathcal{F}(X) & \to & \prod \mathcal{F}(U_i) & \to & \prod \mathcal{F}(U_{ij}) & \\ & \| & & \| & & \| & \\ 0 \to & \mathbb{Z}^3 & \to & \underline{\mathbb{Z}^2} & \to & 0 & \end{array}$$

⇝ sheaf sequence is which cannot be exact.

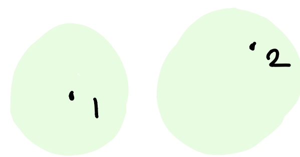
$(0,0,1) \to 0$

$\neq 0 \qquad 0$

1.2 In the notation of Example 1.6, the differential operator gives a map of sheaves $D\colon \mathscr{O}_X \to \mathscr{O}_X$, where as previously $X \subseteq \mathbb{C}$ is an open set. Show that the assignment

$$\mathscr{A}(U) = \{ f \in \mathcal{O}_X(U) \mid Df = 0 \}$$

defines a subsheaf $\mathscr{A}$ of $\mathscr{O}_X$. Show that if U is a connected open subset of X, one has $\mathscr{A}(U) = \mathbb{C}$. In general for a not necessarily connected set U, show that $\mathscr{A}(U) = \prod_{\pi_0 U} \mathbb{C}$ where the product is taken over the set $\pi_0 U$ of connected components of U.



$U \subseteq X$ connected open

$Df = 0 \Rightarrow f$ has zero derivative

$\Rightarrow f$ is locally constant

$\Rightarrow f$ is constant (U connected)

$\Rightarrow A(U) = \mathbb{C}$.

If U has components $\{U_i\}_{i \in I}$, then

we can define $A(U) \longrightarrow \prod_{i \in I} \mathbb{C}$ compatible with restriction maps.

$(f : U \to \mathbb{C}) \mapsto (f(x_i))_i$

where $x_i \in U_i$ is any point

This is clearly an isomorphism $\Rightarrow$ OK.

1.3 Consider the sheaf $C(X, \mathbb{R})$ from Example 1.5 and the subpresheaf $\mathcal{F}$, defined by setting $\mathcal{F}(U) = C_b(U, \mathbb{R})$, the group of *bounded* continuous functions. Show that $\mathcal{F}$ is not a sheaf. What is the saturation of $\mathcal{F}$ in $C(X, \mathbb{R})$?

The Gluing axiom fails:

Consider the covering

$$\mathbb{R} = \bigcup_{r>0} \underbrace{B(0,r)}_{\{|x|<r\}}$$

$\rightsquigarrow$ On $B(0,r)$ $f(x) = x$ is bounded.

But there is no bounded $f : \mathbb{R} \to \mathbb{R}$ s.t

$$f\big|_{B(0,r)} = x \quad \text{for all } r.$$

The saturation is given by

$$\overline{C_b}(U) = \left\{ \underset{\text{continuous}}{f : U \to \mathbb{R}} \;\middle|\; \begin{array}{l} f \text{ locally lies} \\ \text{in } C_b \end{array} \right\}$$

$$= \left\{ \underset{\text{continuous}}{f : U \to \mathbb{R}} \;\middle|\; f \text{ is locally bounded} \right\}$$

$$= C(U, \mathbb{R})$$

(any continuous function is locally bounded).

I $\quad i \geq j$

$$\cdots \to G_i \xrightarrow{\phi_{ij}} G_j \to \cdots \quad \rightsquigarrow \quad G = \lim G_i$$

$$G \longrightarrow G_i$$
$$\downarrow \phi_{ij}$$

$H \longrightarrow G_j$

1.5 Let $G_{i i\in I}$ be a directed system of abelian groups with maps f_{ij}. The projective limit of $\{G_i\}_{i\in I}$ has an explicit construction as the subset of the product $\prod_{i\in I} G_i$ consisting of 'sequences' (x_i) such that $x_j = f_{ij}(x_i)$ for all pairs $i, j \in I$. In other words,

$$\varprojlim G_i = \{(\cancel{x}_i)_{i\in I} \mid x_j = f_{ij}(x_i) \forall i, j \in I\} \subseteq \prod_{i\in I} G_i$$

$j \leq i$

Show this.

We check the universal property:

$$i \geqslant j \qquad \begin{tikzcd} \vdots \arrow[d] & \\ G_i \arrow[d] & G = \varprojlim G_i \arrow[l] \arrow[dl] \\ G_j \arrow[d] & \\ \vdots & \end{tikzcd}$$

s.t

$$\begin{tikzcd} G_i \arrow[d] & H \arrow[l] \arrow[dl] \arrow[dr, "\exists!\text{ factorization}"] & \\ G_j & & G \arrow[ll] \end{tikzcd}$$

x_j is determined by x_i which is "further out".

$$G := \left\{ (\cancel{x}_i)_{i\in I} \;\middle|\; \begin{array}{c} x_j = f_{ij}(x_i) \\ \forall i,j \end{array} \right\} \subseteq \prod_{i\in I} G_i$$

Given $\begin{tikzcd} G_i \arrow[d, "f_{ij}"'] & H \arrow[l, "h_i"'] \arrow[dl, "h_j"] \\ G_j & \end{tikzcd}$ Define $h : H \longrightarrow G$, $y \longmapsto (h_i(y))_{i\in I}$

we have $f_{ij}(h_i(y)) = h_j(y)$ by the diagram

$\Rightarrow (h_i(y)) \in G \rightsquigarrow$ ok.

1.10 Exhibit a directed system in the category sets of finite sets that does not have a direct limit in sets.

★

$G_i = \{0, 1, \dots, i\}$

$i \leq j \rightsquigarrow f_{ij} : G_i \to G_j$ inclusion

$\Rightarrow \varinjlim G_i$ is not a finite set.

If $G = \varinjlim G_i$ is finite, then we get a contradiction by applying the universal property to $H = G_d$ for $\gg 0$:

$$\begin{tikzcd} G_i \arrow[r] \arrow[d] & G_d \\ G_j \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \{0..i\} \arrow[r] \arrow[d] & \{0..d\} \\ \{0..j\} \arrow[ur] & \end{tikzcd}$$

send $d+1, \dots, i$ all to 0.

$$\begin{tikzcd} G_d \arrow[r, "id"] \arrow[d] & G_d \\ G \arrow[ur, "h"] & \end{tikzcd}$$

$\rightsquigarrow$ there can not be any such map h

Exercise 1.11 Check that $(\operatorname{Ker}\phi)_x = \operatorname{Ker}(\phi_x)$ for a morphism of abelian sheaves $\phi : \mathcal{F} \to \mathcal{G}$. ★

$\phi : \mathcal{F} \to \mathcal{G} \quad \rightsquigarrow \quad \phi(V) : \mathcal{F}(V) \longrightarrow \mathcal{G}(V)$

If $s \in \ker \phi(V)$, then $s_x \in \ker \phi_x$ because of:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r] \arrow[d] & \mathcal{G}(V) \arrow[d] \\ \mathcal{F}(W) \arrow[r] & \mathcal{G}(W) \end{tikzcd} \qquad \begin{tikzcd} s \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\ s|_W \arrow[r, mapsto] & 0 \end{tikzcd}$$

Can define

$$\begin{aligned} (\ker \phi)_x &\xrightarrow{\ \Phi\ } \ker \phi_x \\ (s, V) &\longmapsto s_x \end{aligned}$$

Φ is a group homomorphism ✓

Φ is injective: If $[(s,V)] \in (\ker \phi)_x$ maps to zero

$\Rightarrow (s, V) = (0, V)$ for some V $\Rightarrow$ injective.

Φ surjective: Any $s_x \in \ker \phi_x \subseteq \mathcal{F}_x$ is induced by a section (s, V) for some V.
$\phi(s)$ is an element s.t. $\phi(s)_x = 0 \rightsquigarrow$ by shrinking V we may assume $\phi(s) = 0$ on V. $\Rightarrow s \in \ker \phi(V)$
$\Rightarrow s_x$ is induced by (s, V).

$\mathcal{B}$ basis

Definition 1.47 *A $\mathscr{B}$-presheaf $\mathcal{F}$ consists of the following data:*

i) For each $U \in \mathscr{B}$, an abelian group $\mathcal{F}(U)$;

ii) For all $U \subset V$, with $U, V \in \mathscr{B}$, a restriction map $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$.

As before, these are required to satisfy the relations $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ and $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$. A $\mathscr{B}$-sheaf is a $\mathscr{B}$-presheaf satisfying the Locality and Gluing axioms for open sets in $\mathscr{B}$.

Since the intersections $V \cap V'$ of two sets $V, V' \in \mathscr{B}$ need not lie in $\mathscr{B}$, we need to clarify what we mean in the Gluing axiom. Given a covering of $U \in \mathscr{B}$ by subsets $U_i \in \mathscr{B}$ and a covering $\{U_{ijk}\}$ of $U_i \cap U_j$: If $s_i \in \mathcal{F}_0(U_i)$ are sections such that $s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$, then the s_i should glue to an element in $s \in \mathcal{F}_0(U)$.

Proposition 1.52 *Let X be a topological space and let $\mathscr{B}$ be a basis for the topology on X. Then*

i) Every $\mathscr{B}$-sheaf $\mathcal{F}_0$ extends uniquely to a sheaf $\mathcal{F}$ on X.

ii) If $\phi : \mathcal{F}_0 \to \mathcal{G}_0$ is a morphism of $\mathscr{B}$-sheaves, then ϕ extends uniquely to a morphism $\phi : \mathcal{F} \to \mathcal{G}$ between the corresponding sheaves.

iii) The stalk of the extended sheaf $\mathcal{F}$ at a point x equals $\varinjlim_{x \in U,\ U \in \mathscr{B}} \mathcal{F}_0(U)$.

Inverse limits

We can similarly define the *inverse limit* (also called the *projective limit* or just the *limit*) of a directed system $\{M_i\}$. The definition is the same as above, just with the arrows reversed. More precisely, we are given a collection of A-modules M_i, indexed by a directed ordered set I. Moreover, for every pair i, j from I with $i \leqslant j$, we are given an A-linear map $\phi_{ij}\colon M_j \to M_i$ which satisfy the conditions:

- $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$;
- $\phi_{ii} = \mathrm{id}_{M_i}$.

The inverse limit $\varprojlim M_i$ is then a module together with universal maps $\phi_i\colon \varprojlim M_i \to M_i$ satisfying $\phi_j = \phi_i \circ \phi_{ij}$. That is, for any other module N together with maps $\psi_i\colon N \to M_i$ such that $\psi_i = \phi_{ij} \circ \psi_i$ there is a unique A-linear map $\eta\colon N \to \varprojlim M_i$ such that $\psi_i = \phi_i \circ \eta$.

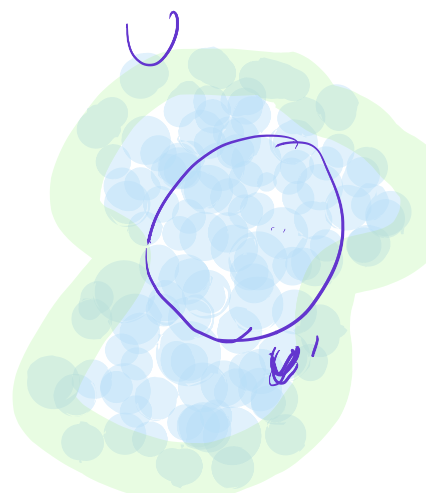
Proposition 1.20 *Every directed inverse system of modules has a limit.*

Proof: Consider the product $\prod_i M_i$ and define a submodule by

$$L = \{\, (x_i) \mid x_i = \phi_{ij}(x_j) \text{ for all pairs } i, j \text{ with } i \leqslant j \,\},$$

The projections induce maps to $\phi_i\colon L \to M_i$, and we claim that L together with these maps constitute the inverse image of the system. A family of maps $\psi_i\colon N \to M_i$ defines map $\eta\colon N \to \prod_i M_i$ by $x \mapsto (\psi_i(x))$ which takes values in L when the ψ_i's satisfying the compatibility constraints $\psi_i = \phi_{ij} \circ \psi_j$. It is clearly unique, and that gives the desired universal property. ❑

$\mathcal{F}_0$ $\mathcal{B}$ knippe $\mathcal{F}_0(V)$ $V \in \mathcal{B}$



PROOF: For clarity, let us denote $\rho^0_{VV'} : \mathcal{F}_0(U) \to \mathcal{F}_0(V)$ the restriction map over subsets $V, V' \in \mathscr{B}$ with $V \supset V'$.

For any open set $U \subset X$, we define $\mathcal{F}(U)$ to be the inverse limit of the $\mathcal{F}_0(V)$, where V runs over the basis of open subsets contained in U. More precisely,

$$\begin{aligned} \mathcal{F}(U) &= \varprojlim_{V \in \mathscr{B}, U \supset V} \mathcal{F}_0(V) \\ &= \left\{ (s_V) \in \prod_{\mathscr{B} \ni V \subset U} \mathcal{F}_0(V) \,\middle|\, \rho^0_{VW}(s_V) = s_W \text{ for all } W \subset V \subset U,\ V, W \in \mathscr{B} \right\} \end{aligned}$$

$\mathcal{F}(U) = \mathcal{F}_0(U)$

$U \in B$

Note that when $U \supset V$ and $V \in \mathscr{B}$, we have a canonical map

$$\pi_{UV} : \mathcal{F}(U) \to \mathcal{F}_0(V)$$

induced by the projection $\prod_{W \subset U, W \in \mathscr{B}} \mathcal{F}_0(W) \to \mathcal{F}_0(V)$ onto the 'V-th' factor. These commute with the restriction maps $\rho^0_{VV'}$ for $V, V' \in \mathscr{B}$ in the sense that $\rho^0_{VV'} \circ \pi_{UV} = \pi_{UV'}$ for all $V' \subset V \subset U$ with $V, V' \in \mathscr{B}$. In particular, we see that there is a canonical isomorphism $\pi_{VV} : \mathcal{F}(V) \to \mathcal{F}_0(V)$ for every $V \in \mathscr{B}$.

The projections $\prod_{\mathscr{B} \ni V \subset U} \mathcal{F}_0(V) \to \prod_{\mathscr{B} \ni V \subset U'} \mathcal{F}_0(V)$ furthermore induce, for each $U \supset U'$, a morphism

$$\rho_{UU'} : \mathcal{F}(U) \to \mathcal{F}(U').$$

By construction, this has the property that $\rho_{UU''} = \rho_{U'U''} \circ \rho_{UU'}$ for each chain of opens $U \supset U' \supset U''$. This means that $\mathcal{F}$ is a presheaf on X which extends $\mathcal{F}_0$. We note that *iii*) follows immediately, since we may compute the inverse limit using subsets of $\mathscr{B}$. Checking that the sheaf axioms and that $\mathcal{F}$ is unique is left as exercises (Exercises 1.18 and 1.19).

For *ii*): $\phi_0 : \mathcal{F}_0 \to \mathcal{G}_0$ being a map of $\mathscr{B}$-sheaves, amounts to saying that the following diagram commutes for each $V \supset V'$ in $\mathscr{B}$:

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi_V"] \arrow[d] & \mathcal{G}(V) \arrow[d] \\ \mathcal{F}(V') \arrow[r, "\phi_{V'}"] & \mathcal{G}(V') \end{tikzcd}$$

Taking the inverse limit over all subsets V contained in U, we obtain a natural map

$$\mathcal{F}(U) = \varprojlim_{V \in \mathscr{B}, V \subset U} \mathcal{F}_0(V) \to \varprojlim_{V \in \mathscr{B}, V \subset U} \mathcal{G}_0(V) = \mathcal{G}(U)$$

which extends ϕ_0. Again this must be unique, as it is completely determined by ϕ_0 on stalks. ❑

Exercise 1.18 Prove that the extension sheaf $\mathcal{F}$ constructed above satisfies the two sheaf axioms. ★

✲ **Exercise 1.19** Prove that the extended sheaf $\mathcal{F}$ is unique.

EXERCISE 1.19 Verify the point (iii) above, that is, show that $\mathcal{F}_x \simeq \varinjlim_{x\in U,\, U\in\mathscr{B}} \mathcal{F}(U)$. ★

$\Phi_V : \mathcal{F}_0(V) \xrightarrow{\sim} \mathcal{F}(V)$

compatible with restrictions:

$$\begin{tikzcd}
\mathcal{F}_0(U) \arrow[r, "\Phi_U"] \arrow[d, "\rho_{UV}"'] & \mathcal{F}(U) \arrow[d, "\rho^V"] \\
\mathcal{F}_0(V) \arrow[r, "\Phi"] & \mathcal{F}(V)
\end{tikzcd}$$

$$\Rightarrow \varinjlim_{\substack{V\in B \\ V\in x}} \mathcal{F}_0(V) \xrightarrow{\sim} \varinjlim_{\substack{V\in B \\ V\in x}} \mathcal{F}(V) \underset{\substack{\uparrow \\ \beta \text{ is} \\ \text{a basis}}}{\simeq} \varinjlim_{V\in x} \mathcal{F}(V) = \mathcal{F}_x$$

Exercise 1.20 Verify the point (ii) above, that is, show that a morphism of $\mathscr{B}$-sheaves $\phi : \mathcal{F} \to \mathcal{G}$ has a unique extension to a map of sheaves. ★