**Exercise 2.1** Let $\mathfrak{a} \subset A$ be an ideal. Show that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$. Hint: If $f \notin \sqrt{\mathfrak{a}}$ the ideal $\mathfrak{a}A_f$ is a proper ideal in the localization $A_f$, hence contained in a maximal ideal. ★

$f \in \sqrt{a} \Rightarrow f^n \in a$ for en $n > 0$

$\Rightarrow f^n \in \mathfrak{p}$ for alle $\mathfrak{p} \supset a$

$\Rightarrow f \in \mathfrak{p}$ for alle $\mathfrak{p} \supset a$

$\Rightarrow f \in \bigcap_{\mathfrak{p} \supset a} \mathfrak{p}$

Følger hintet: $\iota : A \longrightarrow A_f$ lokalisering

Dersom $f \notin \sqrt{a} \Rightarrow a A_f$ propert ideal

$\left( 1 = \sum \frac{a_i}{f^n} \quad a_i \in a \right.$

$\Rightarrow f^n = \sum a_i$

$\left. \Rightarrow f^n \in a \right)$

$\Rightarrow \exists$ maksimalt ideal $m \subset A_f$ som inneholder $a A_f$

$\Rightarrow \mathfrak{p} = \iota^{-1}(m)$ er et prim ideal som ikke inneholder $f$ (men inneholder $a$)

$\Rightarrow f \notin \bigcap_{\mathfrak{p} \supset a} \mathfrak{p}$

**EXERCISE 2.2** Show that $D(f) = \varnothing$ if and only if $f$ is nilpotent. HINT: Use that $\sqrt{(0)} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$. ★

$f$ nilpotent $\Rightarrow$ $f^n = 0$

$\Rightarrow$ $D(f) = D(f^n) = D(0) = \varnothing$

$D(f) = \varnothing$ $\Rightarrow$ $V(f) = X$

$\Rightarrow$ $f \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$

$\Rightarrow$ $f \in \bigcap \mathfrak{p} = \sqrt{0}$

$\Rightarrow$ $f$ nilpotent.

$\ast$ **2.12** *(Direct products of rings)* Assume that $e_1, \ldots, e_r$ is a complete set of orthogonal idempotents in the ring $A$, meaning that one has $1 = e_1 + e_2 + \cdots + e_r$, that $e_i e_j = 0$ when $i \neq j$, and that $e_i^2 = e_i$. Such a set of idempotents corresponds to a decomposition of $A$ as the direct product $A = A_1 \times A_2 \times \ldots \times A_r$ where $A_i = e_i A$ for $i = 1, \ldots, r$ (each $A_i$ is a subring of $A$ with unit element $e_i$).

a) Let $\mathfrak{a}$ be an ideal in $A$ and let $\mathfrak{a}_i = \mathfrak{a} e_i$. Show that $\mathfrak{a}_i$ is an ideal in $A_i$ and that $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2 + \cdots + \mathfrak{a}_r$.

b) Show that $\mathfrak{a}$ is a prime ideal if and only $\mathfrak{a}_i = A_i$ for all but one index $i_0$ and $\mathfrak{a}_{i_0}$ is a prime ideal in $A_{i_0}$.

c) Show that $\operatorname{Spec} A$ is not connected: It holds that $\operatorname{Spec} A = \bigcup_i \operatorname{Spec} A_i$, the union being disjoint and each $\operatorname{Spec} A_i$ being open and closed in $\operatorname{Spec} A$.

a) $\quad \mathfrak{a}_1 = \mathfrak{a} e_1 \quad$ is an ideal:

$xe_1, ye_1 \in \mathfrak{a}e_1 \quad \Rightarrow \quad xe_1 + ye_1 = (x+y)e_1$
$(xe_1)(ye_1) = (xy)e_1 \quad \rightsquigarrow$ subgroup

$xe_1 \in \mathfrak{a}e_1$
$ae_1 \in \mathfrak{a}e_1 \quad \Rightarrow \quad (ae_1)(xe_1) = (ax)e_1 \in \mathfrak{a}e_1 \quad \Rightarrow$ ideal ✓

$\mathfrak{a} \supseteq \mathfrak{a}_1 + \cdots + \mathfrak{a}_r \quad$ ok.

$\subseteq$: $\quad x \in A = A_1 \times \cdots \times A_r \quad \Rightarrow \quad x = \underbrace{xe_1}_{\in A_1} + \underbrace{xe_2}_{\in A_2} + \cdots + \underbrace{xe_r}_{\in A_r}$

b) $\quad \mathfrak{p} \subset A = A_1 \times \cdots \times A_r$

$\mathfrak{p} \neq (1) \quad \Rightarrow \quad$ wlog $\; e_{i_0} \notin \mathfrak{p}$

$\mathfrak{p}_{i_0} \subset A_{i_0}$ is prime: $\quad$ proper ✓

$(ae_{i_0})(be_{i_0}) = (ab)e_{i_0} \in \mathfrak{p}_0$

$\Rightarrow$ either $a \in \mathfrak{p}$ or $b \in \mathfrak{p} \quad \Rightarrow$ ok

If $i \neq i_0$, then

$e_i e_{i_0} = 0 \quad \Rightarrow \quad e_i \in \mathfrak{p}$ (since $\mathfrak{p}$ is prime)

$\Rightarrow \quad \mathfrak{p}_i = A_i$

c) $\quad U_i = \{ \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} A_i \}$ is open: $\quad D(e_i) \quad \mathfrak{p} \not\ni e_i$

also closed: $\quad \mathfrak{p} \in U_i \Leftrightarrow \mathfrak{p} \in A_i \Leftrightarrow \mathfrak{p} \cdot e_j = 0$ for all $j \neq i$

$\Leftrightarrow \mathfrak{p} \in V(I_i) \quad I_i = \{ x \in A \mid xe_j = 0 \}_{j \neq i}$

$\operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A$ induced by $I_i \subset A$

$\rightsquigarrow$ $\varphi_i$ is is a homeomorphism onto $V(I_i) = N_i$.

$e^2 = e$ $\quad$ $\mathfrak{a} = \ker \varphi$ $\quad$ $x \in \mathfrak{a} \Rightarrow x^n = 0$

**2.13** *(Lifting of idempotents)* Let $A \xrightarrow{\varphi} B$ be a surjective map of (~~not necessarily commutative rings~~) rings whose kernel $\mathfrak{a}$ is locally nilpotent; that is, every element of $\mathfrak{a}$ is nilpotent. Let $e$ be an idempotent in $B$. The aim of the exercise is to show that there is an idempotent $f$ in $A$ mapping to $e$. Choose any element $x$ in $A$ that maps to $e$ and let $y = 1 - x$.

a) Show that $xy \in \mathfrak{a}$.

b) Let $n$ be such that $(xy)^n = 0$ and define $f = \sum_{i>n} \binom{2n}{i} x^i y^{2n-i}$ and $g = \sum_{i\leqslant n} \binom{2n}{i} x^i y^{2n-i}$. Show that $1 = f + g$ and that $fg = 0$.

c) Conclude that $f$ is an idempotent in $A$ that maps to $e$.

d) Show that if $\operatorname{Spec} A$ is not connected, the ring $A$ is a non-trivial direct product; that is, $A \simeq B \times C$ for non-zero rings $B$ and $C$

$A \xrightarrow{\varphi} B$ $\quad$ $\mathfrak{a} = \ker \varphi$ $\quad$ locally nilpotent.

$e^2 = e$ idempotent.

let $x \in A$ s.t. $\varphi(x) = e$ $\quad$ $y = 1 - x$

a) $xy = x(1-x) = x - x^2$ maps to $e - e^2 = 0 \Rightarrow xy \in \mathfrak{a}$

b) $f = \sum_{i>n} \binom{2n}{i} x^i y^{2n-i}$ $\quad$ $g = \sum_{i\leq n} \binom{2n}{i} x^i y^{2n-i}$

$1 = (x + (1-x))^{2n} = (x+y)^{2n} = f + g$ by Binomial thm

$fg = 0$ since every monomial is divisible by $(xy)^n$.

c) $f^2 = f(1-g) = f - fg = f$

$\phi(f) = \sum_{i>n} \binom{2n}{i} \phi(x)^i (1 - \phi(x))^{2n-i}$ $\quad$ $(1-e)^2 = 1 - 2e + e^2 = 1 - e$

$= \sum_{i>n} \binom{2n}{i} e^i (1-e)^{2n-i} = \sum_{i>n} \binom{2n}{i} e (1-e) + e$ $\quad$ $i = 2n$

$= \underline{e}$

d) Suppose $X = \operatorname{Spec} A = U_1 \cup U_2$ $\quad U_1 \cap U_2 = \emptyset$

$U_i$ clopen

$U_1 = V(I)$

$U_2 = V(J)$

$\emptyset = U_1 \cap U_2 = V(I+J)$ $\qquad I + J = (1)$

$X = U_1 \cup U_2 = V(I \cap J)$ $\Rightarrow$ $I \cap J \subseteq \sqrt{0}$

Pick $z \in I$, $w \in J$ s.t. $z + w = 1$

$\rightsquigarrow$ $z = z(z+w) = z^2 + zw$ $\Rightarrow$ $z^2 - z \in I \cap J \subseteq \sqrt{0}$

$\bar{z}^2 = \bar{z}$ in $A/\sqrt{0}$

Apply c) to $A/\sqrt{0}$ $\rightsquigarrow$ $\exists\, x \in A$ idempotent mapping to $\bar{z}$

Now we have $A \xrightarrow{\varphi} B \times C$, $\quad a \longmapsto (ax, a(1-x))$ $\qquad B = Ax$, $C = A(1-x)$ $\simeq$ subrings of $A$

$\varphi$ surjective: given $b \in Ax$, $b = a'x$; $c \in A(1-x)$, $c = a''(1-x)$ $\rightsquigarrow$ define $a = a' + a''$.

$\varphi$ injective: $\varphi(a) = (0,0) \Rightarrow ax = 0 = a(1-x) = a - ax = a \Rightarrow a = 0$

$\rightsquigarrow$ done.

## Alternative solution

$X = \operatorname{Spec} A = U_1 \cup U_2$ $\qquad U_i$ open

$U_1 \cap U_2 = \emptyset$

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1 \cap U_2)$$

$\mathcal{O}_X(X) = A$, $\quad \mathcal{O}_X(U_1 \cap U_2) = 0$

$\rightsquigarrow$ $A \simeq \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ ✓

2.15 Let $\{A_i\}_{i\in I}$ be an infinite sequence of non-trivial rings, and let $X$ be the disjoint union of the spectra $\operatorname{Spec} A_i$. Show that $X$ is not homeomorphic to a spectrum of a ring.

$$X = \coprod_{i \in I} \operatorname{Spec} A_i$$

$X$ is not quasicompact, hence not $\simeq \operatorname{Spec} R$.

( $\{\operatorname{Spec} A_i\}$ cover with no finite subcover. )

✳ **2.18** Compute a primary decomposition for the following ideals and describe their corresponding closed subschemes.

*i)* $I = (x^2y^2, x^2z, y^2z)$ in $k[x,y,z]$.

*ii)* $I = (x^2y, y^2x)$ in $k[x,y]$.

*iii)* $I = (x^3y, y^4x)$ in $k[x,y]$.

*iv)* $I = (x, y, x - yz)$ in $k[x,y,z]$

*v)* $I = (x^2 + (y-1)^2 - 1, y - x^2)$ in $k[x,y]$.

★

i) $(x^2y^2, x^2z, y^2z)$

$= (x^2, x^2z, y^2z) \cap (y^2, x^2z, y^2z)$

$= (x^2, y^2z) \cap (y^2, x^2z)$

$= (x^2, y^2) \cap (x^2, z) \cap (y^2, x^2) \cap (y^2, z)$

ii) $I = (x^2y, y^2x) = (x^2, y^2x) \cap (y, y^2x)$

$= (x^2, y^2) \cap (x) \cap (y) \cap (x, y)$

$= (x^2, y^2) \cap (x) \cap (y)$

iii) $(x^3y, y^4x) = (x^3, y^4x) \cap (y)$

$= (x) \cap (x^3, y^4) \cap (y)$

iv) ??

v) $(x^2 + (y-1)^2 - 1, \; y - x^2)$

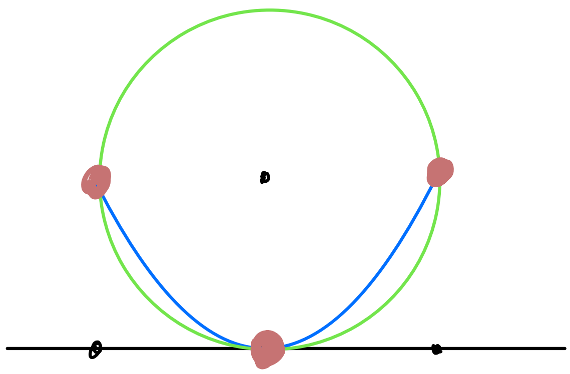
$$= (x^2 + (x^2-1)^2 - 1,\ y - x^2)$$

$$= (x^4 - 2x^2 + x^2,\ y - x^2) = (x^4 - x^2,\ y - x^2)$$

$$= (x^2(x^2-1),\ y - x^2)$$

$$= (x^2, y - x^2) \cap (x-1, y-x^2) \cap (x+1, y-x^2)$$

$$= (x^2, y) \cap (x-1, y-1) \cap (x+1, y-1)$$



$s \in \mathcal{O}_X(U) \rightsquigarrow s(x) \in \mathcal{O}_{X,x}/m_{X,x} =: k(x)$

$a \in A \rightsquigarrow a(x) =$ klassen til $a$ i $A_p/pA_p$
$x = [p]$ $\qquad = k(x)$

**Exercise 3.1** Prove Corollary 3.7.

**Corollary 3.7** *Let $A$ be an integral domain with fraction field $K$, and let $X = \operatorname{Spec} A$. Then $\mathcal{O}_X$ is naturally a subsheaf of the constant sheaf $K_X$, and*

$$\mathcal{O}_X(U) = \left\{ f \in K \;\middle|\; \begin{array}{l} f \text{ can be represented as } g/h \\ \text{where } h(x) \neq 0 \text{ for every } x \in U. \end{array} \right\} \subset K.$$

*Furthermore, we have*

*i)* $\mathcal{O}_X(D(g)) = \{ ag^{-n} \mid f \in A, n \geqslant 0 \} \subset K$;

*ii)* $\mathcal{O}_{X,x} = \{ fg^{-1} \mid f, g \in A, g \notin \mathfrak{p}_x \} \subset K$.

$A_g \longrightarrow A_h \subset K$

$$\mathcal{O}_X(U) = \varprojlim_{D(g) \subset U} A_g$$

$A$ integral domain $\rightsquigarrow$ $A \subset K$ and each $A_g \to A_h$ is injective (inclusion maps in fact!)

$$\rightsquigarrow \varprojlim A_g = \bigcup_{D(g) \subset U} A_g$$

which equals the RHS.

i) $\mathcal{O}_X(D(g)) \overset{\text{def}}{=} A_g = \left\{ \frac{a}{g^n} \;\middle|\; a \in A \right\}$ ok.

ii) $\mathcal{O}_{X,x} = A_{\mathfrak{p}} = \left\{ \frac{a}{g} \;\middle|\; g \notin \mathfrak{p} \right\}$ ok.