Exercise 3.1 Show that being a closed immersion is a property which is local on the target. In clear text: Assume given a morphism $f\colon X \to Y$ and an open cover $\{U_i\}$ of Y. Let $V_i = f^{-1}(U_i)$ and assume that each restriction $f|_{V_i}\colon V_i \to U_i$ is a closed immersion. Prove that then also f is a closed immersion. ★

Exercise 3.2 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. Prove that if both f and g are closed immersions then $g \circ f$ is one as well. ★

3.6 *(The Frobenius morphism)* Let p be a prime number and let A be a ring of characteristic p. The ring map $F_A : A \to A$ given by $a \mapsto a^p$ is called the *Frobenius map* on A.

a) Show that F_A induces the identity map on $\operatorname{Spec} A$;
b) Show that if A is local, then F_A is a local homomorphism;
c) For a scheme X over $\mathbb{F}_p$, define the *Frobenius morphism* $F : X \to X$ by the identity on the underlying topological space and with $F^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$. Show that F_X is a morphism of schemes;
d) Show that F_X is natural in the sense that if $f : X \to Y$ is a morphism of schemes over $\mathbb{F}_p$, we have $f \circ F_X = F_Y \circ f$.

In particular, this exercise shows that for a morphism of schemes $f : X \to Y$, in order to check that f is an isomorphism, is is not enough to check that f is a homeomorphism on the level of topological spaces; also the map $f^\sharp$ must be an isomorphism.

Given a scheme $(X, \mathcal{O}_X)$, and a closed subset $Z \subset X$, we will define a new scheme $(Z, \mathcal{O}_Z)$ with underlying topologicail space Z. The structure sheaf $\mathcal{O}_Z$ will be defined by $\mathcal{O}_X/\mathcal{J}$, where $\mathcal{J}$ is the sheaf of ideals given by

$$\mathcal{J}(U) = \{s \in \mathcal{O}_X(U) \mid s(x) = 0 \text{ for all } x \in U \cap Z\}$$

Note that this is indeed a sheaf: it is a subpresheaf of a sheaf, so Locality comes for free. And given a collection of sections $s_i \in \mathcal{J}(V_i)$ for a covering $\{V_i\}$ of $V \subset Z$ that agree on the overlaps $V_i \cap V_j$, we may glue them together to a section $s \in \mathcal{O}_X$, which has the property that $s(x) = 0$ for all $x \in V$, since this holds locally, and hence $s \in \mathcal{J}(U)$. For an affine scheme $U = \operatorname{Spec} A$, we have $\mathcal{J}(U) = \sqrt{0}$, the nilradical of A, and $\mathcal{O}_Z(U) = A/\sqrt{0}$ (See Exercise 3.7).

3.7 Show that for an affine scheme $U = \operatorname{Spec} A \subset X$, we have $\mathcal{J}(U) = \sqrt{0}$, and $\mathcal{O}_Z(U) = A/\sqrt{0}$. (There is a subtle point here: Remember that there is a sheafification involved in forming quotients).

3.8 Prove that the morphism $r\colon X_{\mathrm{red}} \to X$ is a closed immersion.