Exercise 4.2 Let $\{U_i\}_{i\in I}$ be an open cover of X. Let $\mathscr{B}$ be the collection of open sets V so that $V \subset U_i$ for some i. Show that $\mathscr{B}$ is a basis for the topology, and use this to give another proof of Proposition 4.2. ★

$W \subseteq X$ åpen

$$\Rightarrow \quad W = \bigcup_{i\in I} (W \cap U_i)$$

disse er åpne i X, og ligger i $\mathcal{B}$.

$\rightsquigarrow$ $\mathcal{B}$ er en basis.

$\mathcal{F}_{U_i}$ knipper definert på hver U_i; klart at $\mathcal{F}_{U_i}|_W = \mathcal{F}_{U_j}|_W$ om $W \subset U_i \cap U_j$

$\Rightarrow$ $\mathcal{F}$ er et $\mathcal{B}$-knippe

$\Rightarrow$ utvider entydig til et knippe på X.

$\Rightarrow$ Prop 4.2 følger.

Exercise 4.4 Prove Proposition 4.10

Proposition 4.10 *Let X be a scheme and let K be a field. Then to give a morphism of schemes $\operatorname{Spec} K \to X$ is equivalent to giving a point $x \in X$ plus an embedding $k(x) \to K$.*

$\Rightarrow$: Given $f : \operatorname{Spec} K \longrightarrow X$

$x := f(\operatorname{Spec} K)$

$$\begin{tikzcd} f^\#_x : \mathcal{O}_{X,x} \arrow[r] & f_* \mathcal{O}_{\operatorname{Spec} K} \arrow[d, equal] \\ & K \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[rr] \arrow[dr] & & K \\ & k(x) \arrow[ur] & \end{tikzcd} \quad \rightsquigarrow \quad k(x) \to K$$

$\Leftarrow$ Given $x \in X$, $k(x) \hookrightarrow K$

$$f : \operatorname{Spec} K \longrightarrow X$$
$$* \longmapsto x$$

$$f^\# : \mathcal{O}_X \longrightarrow f_* \mathcal{O}_{\operatorname{Spec} K}$$

over $U \subseteq X$:

$$s \longmapsto \begin{cases} 0 & x \notin U \\ \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow k(x) \longrightarrow K & x \in U \end{cases}$$

$\rightsquigarrow$ ok.

Exercise 5.4 Compute the space $\Gamma(X, \mathcal{O}_X)$ of global sections and describe the canonical map $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$. ★

$(X = Bl_0 \mathbb{A}^2)$

$X = U_1 \cup U_2$ $\qquad U_1 = \operatorname{Spec} k[x, t]$

$U_2 = \operatorname{Spec} k[y, s]$

$$0 \to \Gamma(X, \mathcal{O}_X) \to \begin{array}{c}\Gamma(U_1, \mathcal{O}_{U_1}) \\ \oplus \\ \Gamma(U_2, \mathcal{O}_{U_2})\end{array} \to \Gamma(U_{12}, \mathcal{O}_{U_{12}})$$

$$\begin{array}{c} \| \\ k[x, x^{-1}y] \\ \oplus \\ k[y, y^{-1}x] \end{array} \xrightarrow{\rho} k[x^{-1}y, y^{-1}x]$$

$$\Gamma(X, \mathcal{O}_X) = \left\{ \begin{pmatrix} p(x, x^{-1}y), \\ q(y, y^{-1}x) \end{pmatrix} \;\middle|\; \begin{array}{l} p(x, x^{-1}y) \\ = q(y, y^{-1}x) \end{array} \right\}$$

$$= k[x, y].$$

$\Gamma(X, \mathcal{O}_X) \leftarrow k[x, y]$ indusert av nedblåsningen $p: X \longrightarrow \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$

$$p^\#: \underset{x}{\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)} \longrightarrow \underset{\longrightarrow}{p_* \mathcal{O}_X(\mathbb{A}^2) = \mathcal{O}_X(X)} \;\; x$$

$$Y \hookrightarrow Y$$

Exercise 5.8 Prove Proposition 5.4. (A more general result will be proved in Chapter 15). ★

Proposition 5.4 $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$

$$Z = \mathbb{P}^1_A \xhookrightarrow{i} \mathbb{P}^n_A$$

$$0 \to I_Z \to \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_{\mathbb{P}^1} \to 0$$

$$0 \to \underset{0}{I_Z(\mathbb{P}^n)} \to \mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \to \underset{\substack{\mathcal{O}(\mathbb{P}^1_A) \\ A}}{\mathcal{O}(\mathbb{P}^1)}$$

$$\mathbb{P}^n_A = \bigcup_{i=0}^n \operatorname{Spec} R_i$$

der $R_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$

$\rightsquigarrow$ knippesekvensen

$$0 \to \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \to \prod_{i=0}^n R_i \to \prod_{i,j} R_i \cap R_j$$

$$\| $$

$$\left\{ (s) \in \prod_{i=0}^n R_i \;\middle|\; s_i = s_j \text{ i } R_i \cap R_j \right\}$$

$$s_i = p_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

$$\rightsquigarrow \quad p_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) = p_j\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right)$$

$$\rightsquigarrow \quad p_i \in A \quad \forall i = 0, \ldots, n.$$

$$\rightsquigarrow \quad p_0 = p_1 = \cdots = p_n = a \in A \quad \rightsquigarrow \text{ ok.}$$

Exercise 7.1 Show that if the fibre product exists in the category C, it is unique up to a unique isomorphism. ★

Universalegenskap:

$$\begin{tikzcd}
Z \arrow[d, "\exists!"] \arrow[dd, bend right] \arrow[dr, bend left] & \\
X_1 \times_S X_2 \arrow[d] \arrow[r] & X_2 \arrow[d] \\
X_1 \arrow[r] & S
\end{tikzcd}$$

Gitt $P, Q \in C$ med samme egenskap:

$$\begin{tikzcd}
Q \arrow[d, "\varphi"] \arrow[dd, bend right] \arrow[dr, bend left] & \\
P \arrow[d] \arrow[r] & X_2 \arrow[d] \\
X_1 \arrow[r] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[d, "\psi"] \arrow[dd, bend right] \arrow[dr, bend left] & \\
Q \arrow[d] \arrow[r] & X_2 \arrow[d] \\
X_1 \arrow[r] & S
\end{tikzcd}$$

Morfier $\varphi : Q \longrightarrow P$, $\psi : P \longrightarrow Q$.

Vi har $\varphi \circ \psi = id$, ved unikhet.

$$\begin{tikzcd}
P \arrow[d, "\psi"] \arrow[ddd, bend right] \arrow[dddr, bend left] & \\
Q \arrow[d, "\varphi"] \arrow[dd, bend right] \arrow[ddr, bend left] & \\
P \arrow[d] \arrow[r] & X_2 \arrow[d] \\
X_1 \arrow[r] & S
\end{tikzcd}$$

Tilsvarende for $\psi \circ \varphi = id$

$\rightsquigarrow$ ok.

$$\begin{tikzcd}
P \arrow[r, "\exists!\phi"] \arrow[rr, bend left] \arrow[dr, bend right] & P \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

ϕ automorfi $\Rightarrow \phi = id$ ved unikhet: universalegenskapen

Exercise 7.2

a) Give an example showing that the fibre product does not always exist in the category of manifolds.

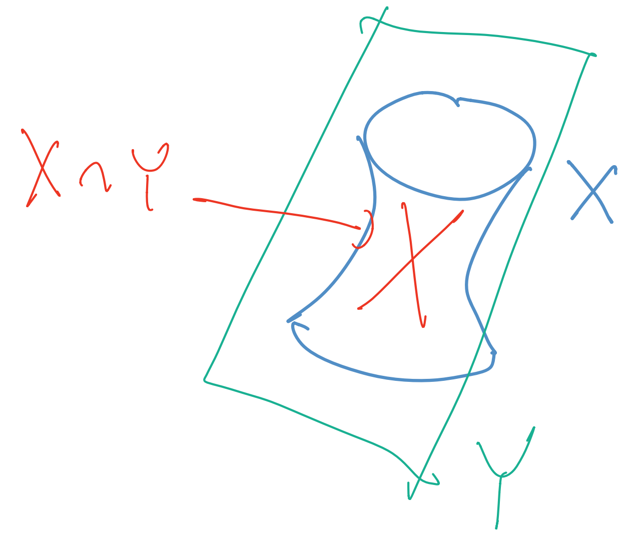
b) Give an example showing that the fibre product does not always exist in the category of affine varieties.

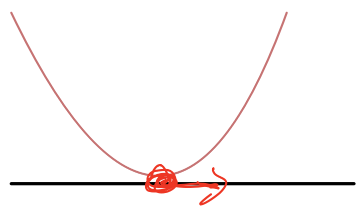
★

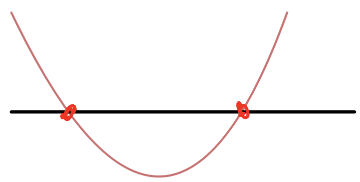
Anta $X, Y \subset Z$

$\rightsquigarrow \quad X \times_Z Y = X \cap Y$ (sjekk universell-egenskapen!)

Lett å finne X, Y s.a $X \cap Y$ er singulær / ikke irredusibel/redusert:







$$\operatorname{Spec} \frac{k[x,y]}{(y-1,\, y-x^2)} \simeq \operatorname{Spec} \frac{k[x]}{(x-1)(x+1)}$$

$$X = \{xy - z = 0\} \subset \mathbb{C}^2$$

$$Y = \{z = 0\} \subset \mathbb{C}^2$$

$$\rightsquigarrow X \cap Y = \{z = xy = 0\} \subset \mathbb{C}^2$$