✲ 8.3 *(Pullback of diagonals)* Let $X \to S$ and $T \to S$ be morphisms between schemes, and as usual, let $X_T = X \times_S T$. Show that the diagonal $\Delta_{X/S}$ pulls back to the diagonal $\Delta_{X_T/T}$; in other words, that there is a canonical Cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{\Delta_{X_T/T}} & X_T \times_T X_T \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

✻ 8.4 *(The diagonal of monomorphisms)* In a general category the classical definition of injective maps is not meaningful and is replaced by the notion of monomorphisms, which reads as follows: a *monomorphism* in the category C is an arrow $f\colon X \to Y$ in C such that for any two arrows $g, g'\colon Z \to X$ in C an equality $f \circ g = f \circ g'$ implies $g = g'$. The dual concept is that of *epimorphisms*: if $g, g'\colon Y \to Z$ are two arrows in C so that $g' \circ f = g \circ f$, then $g = g'$.

In any category were fiber products exist, one has notion of the *graph* of an arrow $f\colon X \to Y$ over S, namely the arrow $\Gamma_f\colon X \to X \times_S Y$ with defining prorty that $\pi_X \circ \Gamma_f = \mathrm{id}_X$ and $\pi_Y \circ \Gamma_f = f$.

a) Show that the diagonal $\Delta_{X/S}$ of any scheme X over S is a monomorphism. As is the graph $\Gamma_f\colon X \to X \times_S Y$ of any morphism $f\colon X \to Y$ between schemes over S.

b) Show that the diagonal of a monomorphism is the identity. In precise terms this means that the following diagram is Cartesian:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & S. \end{array}$$

c) Conclude that "the diagonal of the diagonal" is the identity.

8.6 *(The graph of a morphism)* A morphism $\phi\colon X \to Y$ over S has a *graph* $\Gamma_\phi\colon X \to X \times_S Y$; it is the pullback of the diagonal $\Delta_{Y/S}$ under the morphism $\phi \times \mathrm{id}_Y\colon X \times Y \to Y \times_S Y$. Show that the graph is a closed immersion when Y is separated.

8.16 Show that $X \to S$ is separated if and only if the image of the diagonal map Δ is a closed subset of $X \times_S X$.

✲ **Exercise 9.2** Let R be a graded ring and $\mathfrak{p}$ a homogeneous ideal in R. Show that $\mathfrak{p}$ is prime if and only if $x \cdot y \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ for all homogeneous elements x and y. ★

Exercise 9.3 Let R and S be graded rings and $\phi\colon R \to S$ a homomorphism of graded rings. Show that the inverse image $\phi^{-1}(\mathfrak{p})$ of an ideal $\mathfrak{p} \subset S$ is homogeneous whenever $\mathfrak{p}$ is. ★