Exercise 9.8 Give examples of a non-noetherian graded ring R such that Proj R is Noetherian, of an R that is not of finite type over a field k, but Proj R is and an R which is not an integral domain, but whose projective spectrum Proj R is integral. Hint: The irrelevant ideal is irrelevant. ★

$$\operatorname{Spec} R - V(R_+) \to \operatorname{Proj} R$$

$$S = k[t, x_0, x_1, \ldots]$$

$$\mathfrak{a} = (t x_i,\ x_i x_j \mid i,j \geq 0) \ni x_0^2, \ldots$$

$$R = S/\mathfrak{a} \quad \rightsquigarrow X = \operatorname{Proj} R$$

$$\operatorname{Spec} k = \operatorname{Proj} k[t]$$

Note: X covered by:

$$D_+(t) = \operatorname{Spec}\left(\left(k[t, x_0, x_1, \ldots]/(tx_i,\ x_i x_j)\right)_{(t)}\right)$$

$$= \operatorname{Spec}\left(\frac{k[t, x_0, \ldots]_{(t)}}{(x_i,\ x_i x_j)}\right) = \operatorname{Spec} k.$$

$$D_+(x_i) = \operatorname{Spec}\left(\frac{k[t, x_0, \ldots]_{(x_i)}}{(t x_j,\ x_k x_\ell)} \ni x_i^2\right)$$

$$= \operatorname{Spec}\left(\frac{k[t, x_0, \ldots]_{(x_i)}}{1}\right) = \operatorname{Spec} 0 = \emptyset$$

$\therefore \operatorname{Proj} R = \operatorname{Spec} k \leftarrow$ noetherian + int etc.

Exercise 9.10 In this exercise A denotes a ring. Consider the homomorphism of graded rings $\phi\colon A[x_0, x_1, x_2] \to A[x_0, x_1, x_2]$ defined by the three assignments $x_i \to x_j x_k$ where the indices abide to the rule that $\{i, j, k\} = \{1, 2, 3\}$. Determine the open set $G(\phi)$ in the two cases

a) $A = k$ is a field;

b) A is the ring of integers; *i.e.* $A = \mathbb{Z}$.

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$$S = A[x_0, x_1, x_2] = R$$

$$\Phi : \operatorname{Proj} S \longrightarrow \operatorname{Proj} R \qquad \phi : R \longrightarrow S$$

$$G(\phi) = \{ \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{p} \not\supseteq \phi(R_+) \}$$

a) $R_+ = (x_0, x_1, x_2)$

$$\phi(R_+) = (x_1 x_2, x_0 x_2, x_0 x_1)$$

$$= (x_0, x_1) \cap (x_0, x_2) \cap (x_1, x_2) \quad \leftarrow \text{these are } \underline{\text{maximal}} \text{ in } \operatorname{Proj} S$$

$$\mathfrak{p} \not\supseteq \phi(R_+) \iff \mathfrak{p} \neq (x_0, x_1), (x_0, x_2), (x_1, x_2)$$

b) $V(\phi(R_+)) = V(x_0, x_1) \cup V(x_0, x_2) \cup V(x_1, x_2)$

$\rightsquigarrow$ $G(\phi)$ = complement of $\nearrow$

EXERCISE 9.14 Let $R = k[x,y,z]$ be the polynomial ring given the grading $\deg x = 1$, $\deg y = 2$ and $\deg z = 3$, and let $X = \operatorname{Proj} R$ (also denoted $\mathbb{P}(1,2,3)$). The aim of the exercise is to describe the three covering distinguishes subschemes $D_+(x)$, $D_+(y)$ and $D_+(z)$

a) Show that $R_{(x)} = k[yx^{-2}, zx^{-3}]$ and that $D_+(x) \simeq \mathbb{A}^2_k$. $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

b) Show that $R_{(y)} \simeq k[x^2y^{-1}, z^2y^{-6}, xzy^{-2}]$. Show that the graded algebra homomorphism $k[u,v,w] \to R_{(y)}$ given by the assignments $x \mapsto yx^{-2}$, $v \mapsto z^2y^{-6}$ and $w \mapsto xzy^{-2}$ induces an isomorphism $k[u,v,w]/(w^2 - uv) \simeq R_{(y)}$. Hence $D_+(y)$ is a hypersurface in $\mathbb{A}^3_k$; the so-called "cone over a quadric". Show it is not isomorphic to $\mathbb{A}^2_k$ (check the local ring at the origin).

c) Show that $R_z = k[x^3z^{-1}, y^3z^{-2}, xyz^{-1}]$ and that the map $k[u,v,w] \to R_{(z)}$ defined by the assignments $x \mapsto x^3z^{-1}$, $v \mapsto y^3z^{-2}$ and $w \mapsto xyz^{-1}$ induces an isomorphism $k[u,v,w]/(w^3 - uv) \simeq R_{(z)}$. Show that it is not isomorphic to $\mathbb{A}^2$.

d) Show that the map $R_+ \to k[U,V,W]$ sending $x \mapsto U$, $y \to V^2$ and $z \mapsto W^3$ induces a map $\mathbb{P}^2_k \to \operatorname{Proj} R$, and describe the fibres over closed points.

a) $R_{(x)} = k[x,y,z]_{(x)}$ is generated as a k-vector space by monomials $x^a y^b z^c$ with

$a + 2b + 3c = 0$ $\quad b, c \geq 0$

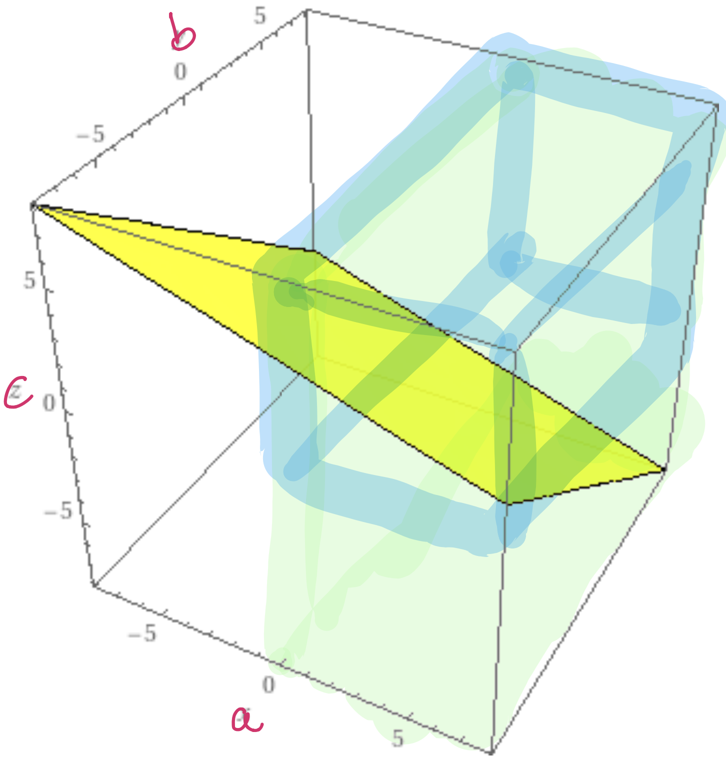
$\longrightarrow \quad a = -2b - 3c$

$\therefore \quad x^a y^b z^c = x^{-2b-3c} y^b z^c = (x^{-2}y)^b (x^{-3}z)^c.$

$\therefore$ any element of $R_{(x)}$ is a polynomial in $x^{-2}y$, $x^{-3}z$,

$u = x^{-2}y$ $\quad v = x^{-3}z$ are alg. indep. $\Rightarrow$ $\operatorname{Spec} R_{(x)} \simeq \mathbb{A}^2$.

b) $R_{(y)}$ generated by $x^a y^b z^c$ s.t. $\begin{cases} a+2b+3c=0 \\ a, c \geq 0 \end{cases}$



given (a,b,c)

$a=0$: $2b+3c=0 \rightsquigarrow (a,b,c) = (0, -3k, 2k)$

$\rightsquigarrow y^{-3} z^2$

$b=0$: $(0,0,0)$ only solution

$c=0$ $a+2b=0 \rightsquigarrow (a,b,c) = (-2b, b, 0)$ $b \leq 0$

$\rightsquigarrow x^2 y^{-1}$

$\therefore$ given (a,b,c) $a+2b+3c=0$ $a,c \geq 0$

$\rightsquigarrow$ can subtract $(2,-1,0)$ to make $a = 0, 1$.

$a=1: \quad 1+2b+3c=0 \quad \rightsquigarrow \quad 1+3c = 2k$ say $\qquad b \equiv 1 \bmod 3$

$\rightsquigarrow \quad b = -k$

$c = -\frac{1}{3}(1+2b) = -\frac{1}{3}(1-2k) = \frac{2k-1}{3}$

$0 \equiv 1+2b+3c$

$\equiv 1-2k \bmod 3$

$= 1+k \bmod 3$

$\therefore k = 3d+2$

$k=2: \qquad x y^{-2} z$

$k=5: \qquad x y^{-5} z^3 = x y^{-2} z \left(y^{-3} z^2\right)$

$\vdots$

$k = 3d+2 \rightsquigarrow c = \frac{6d+4-1}{3} = 2d+1$

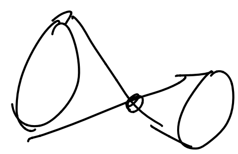
$$x y^{-3d-2} z^{2d+1} = \left(x y^{-2} z\right)\left(y^{-3} z^2\right)^d$$

$$R_{(y)} = k[\underbrace{x y^{-2} z}_{w},\ \underbrace{y^{-3} z^2}_{v},\ \underbrace{x^2 y^{-1}}_{u}]$$

$$w^2 = x^2 y^{-4} z^2 = \left(x^2 y^{-1}\right)\left(y^{-3} z^2\right) = uv$$

$$\therefore \quad R_{(y)} \simeq k[u,v,w]/(w^2 - uv)$$

cone.



not iso to $\mathbb{A}^2$ because local ring is not a UFD.

c) $R_{(Z)}$: $a+2b+3c=0$ $\qquad a,b\geq 0$

$a=0$: $\quad y^3z^{-2}$ $\qquad (a,b,c)=(0,3,-2)$ $\quad \therefore$ wlog $b=0,1,2,$

$b=0$; $\quad x^3z^{-1}$ $\qquad (a,b,c)=(3,0,-1)$ $\quad$ wlog $a=0,1,2$

$c=0$: $\quad (0,0,0)$ only solution.

need to consider $a,b\in\{1,2\}$

$a=1$: $\quad 1+2b+3c=0 \quad \rightsquigarrow \quad b=\cancel{1}$ mod 3

$\rightsquigarrow b=1$

$(a,b,c)=(1,1,-1)$

$a=2$: $\quad 2+2b+3c=0 \quad \rightsquigarrow \quad b=2$ mod 3

$\rightsquigarrow b=2$

$(a,b,c)=(2,2,-2)$

$=2(1,1,-1).$

$$\therefore \quad R_{(Z)}=k[\underset{u}{x^3z^{-1}},\underset{v}{y^3z^{-2}},\underset{w}{xyz^{-1}}]$$

$$w^3=x^3y^3z^{-3}$$

$$=(x^3z^{-1})(y^3z^{-2})=uv$$

$$D_+(z)=\mathrm{Spec}\ \frac{k[u,v,w]}{(w^3-uv)}$$

singular cubic surface

$$\mathbb{P}^2 \xrightarrow{\Phi} \operatorname{Proj} R$$

induced by $k[x,y,z] \longrightarrow k[u,v,w]$

$$x \longmapsto u$$
$$y \longmapsto v^2$$
$$z \longmapsto w^3$$

over $D(u)$:

$$\begin{array}{ccc} D_+(x) & \longrightarrow & D_+(u) \\ \cong & & \| \\ \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \end{array}$$

$$k[x^{-2}y, x^{-3}z] \to k[\tfrac{v}{u}, \tfrac{w}{u}]$$

$$\frac{y}{x^2} \longrightarrow \frac{v^2}{u^2} = \left(\frac{v}{u}\right)^2$$

$$\frac{z}{x^3} \longrightarrow \left(\frac{w}{u}\right)^3$$

coincides with $(x,y) \mapsto (x^2, y^3)$

this is a finite map of degree 6

fiber over, say (a,b) consists of solutions to

$x^2 = a$ $\rightsquigarrow$ two sols if $a \neq 0$

$y^3 = b$ $\rightsquigarrow$ 3 sols if $b \neq 0$.

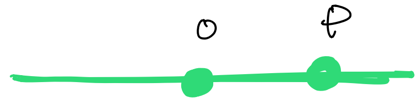
$a = 0$, $b \neq 0$ $\rightsquigarrow$ scheme-theoretic fiber is non-reduced.

✲ 10.2 Find an example of two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ so that the presheaf (10.1) is not a sheaf.

$$(\mathcal{F} \otimes'_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U). \tag{10.1}$$

Let $X = \operatorname{Spec} k[x]$ $\qquad O = \text{origin} \qquad P \neq O$

$$\mathcal{F}(U) = \begin{cases} 0 & O \in U \\ \mathcal{O}_X(U) & \text{otherwise} \end{cases}$$



$$\mathcal{G}(U) = \begin{cases} k & P \in U \\ 0 & \text{otherwise} \end{cases}$$

<u>Claim</u> $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \overset{\varphi}{\cong} \mathcal{G}$ (as sheaves)

$$\varphi'_U : \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \longrightarrow \mathcal{G}(U)$$
$$s \otimes t$$

if $U \ni O$ this map is $0 \otimes \mathcal{G}(U) \longrightarrow \mathcal{G}(U)$ (zero map)

$U \not\ni O$ this is $\mathcal{O}_X(U) \otimes_{\mathcal{O}_X} \mathcal{G}(U) \xrightarrow{\sim} \mathcal{G}(U)$ (isomorphism)

$\rightsquigarrow$ map of presheaves $\rightsquigarrow$ sheafify to get φ

φ is o on stalks:

if $U \not\ni P \rightsquigarrow$ both sides are zero $\rightsquigarrow$ iso

$U \ni P \rightsquigarrow$ both sides are $k \rightsquigarrow k \xrightarrow{\sim} k$

Note that $\mathcal{G}(X) = k$ so $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(X) = k$

However the presheaf has no global sections:

$$\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{G}(X) = 0 \otimes_{\mathcal{O}_X(X)} k = 0.$$

EXERCISE 10.10 Prove that applying f^* commutes with taking tensor products of sheaves, i.e., $f^*(\mathcal{G} \otimes \mathcal{H}) \simeq f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{H}$ for any two $\mathcal{O}_X$-modules $\mathcal{G}$ and $\mathcal{H}$. Does the same hold for f_*? ★

$f^*(\mathcal{G} \otimes \mathcal{H})$ = sheafification of $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\mathcal{G} \otimes \mathcal{H})$

on the level of presheaves:

$$f^{-1}(\mathcal{G} \otimes \mathcal{H}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

$$= \left(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{H}\right) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

$$= \left(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X\right) \otimes_{f^{-1}\mathcal{O}_Y} \left(f^{-1}\mathcal{H} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X\right)$$

$\rightsquigarrow$ OK by sheafifying.

$$f_*(\mathcal{G} \otimes \mathcal{H}) \neq f_*\mathcal{G} \otimes_{\mathcal{O}_Y} f_*\mathcal{H}$$

$$X = \operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k$$

$$f_*(\mathcal{O}_X \otimes \mathcal{O}_X) = f_*\mathcal{O}_X = k[x]$$

$$f_*\mathcal{O}_X \otimes_k f_*\mathcal{O}_X = k[x] \otimes_k k[x] \neq k[x]$$