10.23 Let $\mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ and consider the *twisted cubic curve* C given by the ideal

$$I = (y - x^2, z - x^3)$$

Let $\pi : C \to \mathbb{A}^1_k = \operatorname{Spec} k[z]$ be the projection ~~from~~ to the line $L = V(x,y)$.

i) Show that π is a finite morphism;

ii) Compute $\pi_* \mathcal{O}_C$, $\pi^* \mathcal{O}_{\mathbb{A}^1_k}$ and $\pi^* \mathcal{J}$ where $\mathcal{J}$ is the ideal sheaf of the closed point $0 \in \mathbb{A}^1_k$.

$k[z] \to k[x,y,z]$

$\rightsquigarrow \operatorname{Spec} k[x,y,z] \to \operatorname{Spec} k[z]$

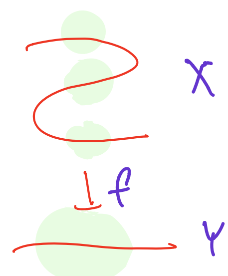
i) $$\underbrace{\frac{k[x,y,z]}{(y-x^2, z-x^3)}}_{= B} \simeq k[z] \oplus k[z]x \oplus k[z]x^2$$

(as a $k[z]$-module) $\rightsquigarrow$ finite.

$f_* \mathcal{O}_X|_U \simeq \mathcal{O}_U^d$

ii) $$\pi_* \mathcal{O}_C = \pi_* \widetilde{B} = \widetilde{k[z]} \oplus \widetilde{k[z]}x \oplus \widetilde{k[z]}\, x^2$$

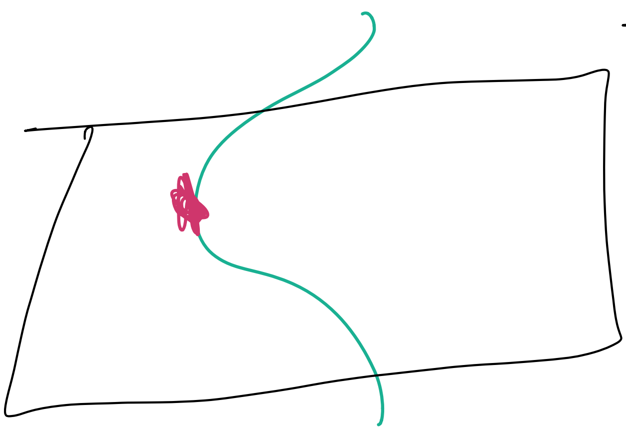
$$\simeq \mathcal{O}_{\mathbb{A}^1}^{3}$$



$$\pi^* \mathcal{O}_{\mathbb{A}^1} = \mathcal{O}_C$$

$$\pi^* \mathcal{J} = \pi^* \widetilde{(z)} = \widetilde{(z) \otimes_{k[z]} k[x,y,z]/(y-x^2, z-x^3)}$$

$$= \widetilde{(z)} \quad \text{in} \quad \frac{k[x,y,z]}{(y-x^2, z-x^3)}$$



$z = 0$

$(y - x^2, x^3)$

10.30 Let $X = Spec(k[T]) = \mathbb{A}^1_k$ and consider the origin $O \in X = \mathbb{A}^1_k$ correponding to the maximal ideal $(T) \subset k[T]$. Define $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ by

$$\begin{cases} \mathcal{I}(U) = \mathcal{O}_X(U) \text{ if } O \notin U \\ \mathcal{I}(U) = 0 \text{ if } O \in U \end{cases}$$

a) Show that $\mathcal{I}$ is an ideal sheaf, and $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I})$ is not a closed subset of X.

b) Show directly that $\mathcal{I}$ is not quasi-coherent.

a) Clear that $\mathcal{I}(U)$ is an ideal for each U ✓

$$\mathrm{Supp}\left(\mathcal{O}/\mathcal{I}\right) = \left\{ \mathfrak{p} \in \mathrm{Spec}\, k[T] \;\middle|\; \left(\mathcal{O}/\mathcal{I}\right)_{\mathfrak{p}} \neq 0 \right\}$$

$$= \mathbb{A}^1 \setminus O \quad \text{open.}$$

b) $\mathcal{I}(X) = 0$

$U = D(f) \Rightarrow \mathcal{I}(U) = \mathcal{O}_X(U)$

$f \notin (x)$

$$\neq \mathcal{I}(X)_f = 0.$$

Exercise 12.1 Let k be a field and let $R = k[x_0, \dots, x_n]$. Let $\pi : \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n_k = \operatorname{Proj} R$ denote the 'quotient morphism' from Exercise 9.9. Show that for a graded R-module M, we have

$$\pi_*(\widetilde{M}|_{\mathbb{A}^{n+1}_k - 0}) = \bigoplus_{n \in \mathbb{Z}} \widetilde{M}(d)$$

$$\begin{array}{c} \mathbb{A}^{n+1}_k - 0 \\ \downarrow \pi \\ \mathbb{P}^n_k \end{array}$$

assosierte graderte modulen til $\widetilde{M}$.

På $U_i = D_+(x_i)$:

$$\rightsquigarrow \pi^{-1}(U_i) = D(x_i) \subset \mathbb{A}^{n+1}$$

$$\rightsquigarrow \pi_*\left(\widetilde{M}|_{\mathbb{A}^{n+1}-0}\right)\Big|_{U_i}$$

$$\begin{array}{ccc} \pi^{-1} U_i & \longrightarrow & U_i \quad \text{affin} \\ \| & & \| \\ \operatorname{Spec} k[x_0 \dots x_n]_{x_i} & \longrightarrow & \operatorname{Spec} k\left[\frac{x_0}{x_i} \dots \frac{x_n}{x_i}\right] \end{array}$$

$$= \pi_*\left(\widetilde{M}|_{D(x_i)}\right)$$

$$= \widetilde{M_{x_i}}$$

where M_{x_i} is regarded as a module over $\left(k[x_0 \dots x_n]_{x_i}\right)_0 =: R_i$

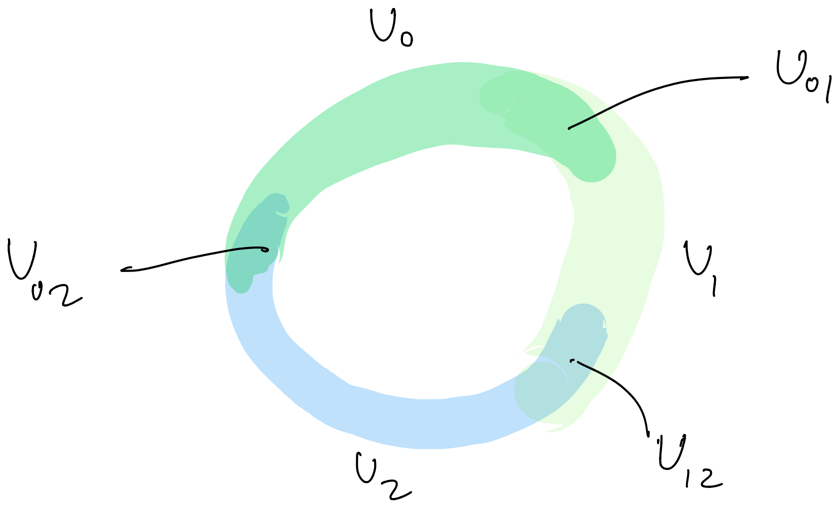
$$= \bigoplus_{d \in \mathbb{Z}} \widetilde{(M(d))}\Big|_{D_+(x_i)}$$

Clear that this isomorphism glues.

EXERCISE 13.1 Let $X = S^1$ and let $\mathcal{U}$ be the covering of X with three pairwise intersecting open sets with empty quadruple intersection. Show that the Čech -complex is of the form

$$\mathbb{Z}^3 \xrightarrow{d^0} \mathbb{Z}^3 \to 0$$

Compute the map d^0 and use it to verify again that $H^i(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 1$ as above. ★



$$C^0 = \mathbb{Z}_{U_0} \oplus \mathbb{Z}_{U_1} \oplus \mathbb{Z}_{U_2} \simeq \mathbb{Z}^3$$

$$\rightsquigarrow 0 \to \mathbb{Z}^3 \xrightarrow{d} \mathbb{Z}^3 \to 0$$

$$C^1 = \mathbb{Z}_{U_{01}} \oplus \mathbb{Z}_{U_{02}} \oplus \mathbb{Z}_{U_{12}} \simeq \mathbb{Z}^3$$

$$\sigma = (a_0, a_1, a_2) \in C^0 \rightsquigarrow d\sigma = \begin{array}{ccc} 01 & 02 & 13 \\ (a_1 - a_0, & a_2 - a_0, & a_2 - a_1) \end{array}$$

$$\mathbb{Z}^3 \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \qquad \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(d\sigma)_{ij} = \sigma_j - \sigma_i$$

$$\ker d^0 = \{(a, a, a)\} \subset \mathbb{Z}^3$$

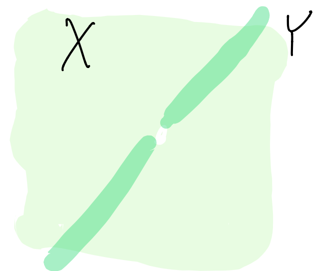
$$\operatorname{coker} d^0 = \frac{\mathbb{Z}f_0 \oplus \mathbb{Z}f_1 \oplus \mathbb{Z}f_2}{\begin{matrix} f_0 - f_2 = 0 \\ f_1 - f_2 = 0 \end{matrix}} \simeq \underline{\mathbb{Z}}.$$

14.3 Let X be a noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X.

(i) Show that $\Gamma(X,\mathcal{F})$ is a finitely generated module over $\mathcal{O}_X(X)$.

(ii) Show more generally that each cohomology group $H^i(X,\mathcal{F})$ is finitely generated over the ring $\mathcal{O}_X(X)$.

This is false! 



$X = \mathbb{A}^2 - 0 \subset \operatorname{Spec} k[x,y]$

$Y = \mathbb{A}^1 - 0 \xhookrightarrow{i} X$ $\qquad y=0$

$\rightsquigarrow \quad \Gamma(X,\mathcal{O}_X) = k[x,y]$

$\Gamma(X, i_*\mathcal{O}_Y) = k[x,x^{-1}]$ $\qquad$ not f.g. ! as $k[x,y]$-module.

$\mathcal{F} = i_*\mathcal{O}_Y$

Men ok dersom X er projektiv + endelig type $/k$.

$\mathcal{F}$ qc $\rightsquigarrow i_*\mathcal{F} = \widetilde{M}$ på $\mathbb{P}^N_k$

$$0 \to \cdots \to R(-a_3)^{b_3} \to R(-a_2)^{b_2} \to R(a_1)^{b_1} \to M \to 0$$

Hilbert Syzygy teorem

$$R(-a_2)^{b_2} \to K \to R(a_1)^{b_1}, \quad K \to 0$$

$\rightsquigarrow H^i(X,\mathcal{F})$ relatert til $H^i(\mathbb{P}^n, \mathcal{O}(-d))$

ved l.e.s. $\rightsquigarrow$ end. gen.

$$0 \to K \to R^b \to M \to 0$$

$$0 \to K \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(-a_i)^{b_i} \to \mathcal{G} \to 0 \qquad (\mathcal{G} = i_* \mathcal{F})$$

end.gen.

end.gen. $\quad H^i$

end.gen. $\quad$ end.gen.

endgen $\quad$ end.gen.

$\vdots \quad \vdots$

$\rightsquigarrow$ induksjon på $\vdots \quad \vdots$ $\quad H^i(\mathbb{P}^n, \mathcal{F}) \implies$ ok.

$(i \geq n+1 \rightsquigarrow H^i(\mathbb{P}^n, \mathcal{F}) = 0)$

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}^n, i_* \mathcal{F})$$

$\parallel$

kohom til Cech komplekset

$$C^i = \prod_{i_0 \ldots i_p} \Gamma(U_{i_0 \ldots i_p}, \mathcal{F}) \qquad U_{i_0 \ldots i_p} \subset X$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad X \cap V_{i_0 \ldots i_p}$

$$H^i(\mathbb{P}^n, i_* \mathcal{F})$$

$\parallel$

Kolum til

$$C'^{i} = \prod_{i_0 \ldots i_p} \Gamma(V_{i_0 \ldots i_p}, i_* \mathcal{F})$$

$$\qquad\qquad\qquad\qquad \parallel$$

$$\qquad\qquad\qquad \Gamma(V_{i_0 \ldots i_p} \cap X, \mathcal{F})$$

$$= C^{i}.$$