MANDATORY ASSIGNMENT MAT4215 – SPRING 2022

The assignment must be submitted electronically by **14:30, Thursday May 12th**.
You need at least 50% to pass. If you have any questions or comments about the problems, feel free to email me at `johnco@math.uio.no`.
Feel free to use whatever results from Commutative Algebra you want.

Problem 1. [10%]
Give an example of a scheme X, a field K and a morphism of ringed spaces $\operatorname{Spec} K \to X$ which is *not* a morphism of schemes.

Problem 2. [10%] For each of the following rings A, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type:

(1) $\mathbb{Z}[i]$
(2) $\mathbb{Z}[\frac{1}{p}]$
(3) $\mathbb{Z}_{(p)}$
(4) $\mathbb{Z} \times \mathbb{Z}$
(5) $\mathbb{Z}[x]$

Problem 3. [30%]
Describe the scheme theoretic fibers in all points of the following morphisms.

(1) $f : \operatorname{Spec} \mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec} \mathbb{C}[x]$
(2) $f : \operatorname{Spec} \mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec} \mathbb{C}[x]$
(3) $f : \operatorname{Spec} \mathbb{C}[x,y]/(xy) \to \operatorname{Spec} \mathbb{C}[x]$
(4) $f : \operatorname{Spec} \mathbb{Z}[x,y]/(xy^2-m) \to \operatorname{Spec} \mathbb{Z}$, where m is a non-zero integer.

Which fibers are irreducible? Which are reduced?

Problem 4. [25%]
Show the following:
(i) The skyscraper sheaf of $\mathbb{C}$ on $\mathbb{A}^1_{\mathbb{C}}$ at the origin 0 is quasi-coherent.
(ii)* The skyscraper sheaf of $\mathbb{C}(T)$ at the origin $p \in \mathbb{A}^1_{\mathbb{C}}$ is an abelian sheaf on X. Show that this can be made into an $\mathscr{O}_X$-module by viewing it as $i_*(\mathbb{C}(T))$, where $i : p \to X$ is the inclusion, and p is given the locally ringed space structure $(p, \mathscr{O}_{X,p})$. Show then that this sheaf is *not* quasi-coherent.
(iii) If X is variety over an algebraically closed field k, then the skyscraper sheaf of k at any closed point $x \in X$ is quasi-coherent.

Problem 5. [25%] Let $\mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ and consider the *twisted cubic curve* C given by the ideal

$$I = (y - x^2, z - x^3)$$

Let $\pi : C \to \mathbb{A}^1_k = \operatorname{Spec} k[z]$ be the projection from the line $L = V(x,y)$.
(i) Show that π is a finite morphism.
(ii) Compute $\pi_*\mathscr{O}_C$, $\pi^*\mathscr{O}_{\mathbb{A}^1_k}$ and $\pi^*\mathscr{J}$ where $\mathscr{J}$ is the ideal sheaf of the closed point $0 \in \mathbb{A}^1_k$.

Date: May 10, 2022.