This is the fifth set of exercises, based on the material in Chapter III, as well as material in Chapter II (concerning Frobenius morphisms).

(1) Let E be an elliptic curve defined over a field K. Let ω be the invariant differential associated to a Weierstrass equation of E. Let $Q \in E$ be a point different from the identity point O_E, with affine coordinates $(x(Q), y(Q))$. Consider the "translation by Q" morphism

$$\tau_Q : E \to E$$

and the induced map of differentials

$$\tau_Q^* : \Omega_E \to \Omega_E.$$

Show that $\tau_Q^* \omega = a_Q \cdot \omega$, and that there exists a rational function

$$a = a(x, y) \in \overline{K}(E)$$

such that $a_Q = a(x(Q), y(Q))$. (You may assume that $\mathrm{char}(K) \neq 2, 3$ so that the Weierstrass equation is as simple as possible.)

(2) Let E be an elliptic curve defined over a field K of characteristic $\mathrm{char}(K) = p$, for a prime p. Let $q = p^r$ for some integer $r \in \mathbb{N}$. Let

$$\phi_q : E \to E^{(q)}$$

be the qth Frobenius morphism. Then $E^{(q)}$ is a cubic curve in $\mathbb{P}^2$ given by a Weierstrass equation. Prove that

$$\Delta(E^{(q)}) = \Delta(E)^q$$

and that

$$j(E^{(q)}) = j(E)^q.$$

Show also that $E^{(q)}$ is again an elliptic curve.

(3) We keep the assumptions and notation from (2), but assume in addition that $K = \mathbb{F}_p$, so that $E = E^{(q)}$. Show that we can factor the qth Frobenius ϕ_q as

$$E \to E \to \ldots \to E$$

(r arrows) where each map is ϕ_p, the pth Frobenius. In other words, show that $\phi_q = \phi_p \circ \ldots \circ \phi_p$ (composition r times). We often write this as $\phi_q = (\phi_p)^r$.

(4) We keep the assumptions and notation from (2) and (3). For any endomorphism

$$\psi : E \to E$$

we say that $Q \in E$ is a *fixed point* of ψ if $\psi(Q) = Q$.

What are the fixed points of ϕ_p? What are the fixed points of ϕ_q, where $q = p^r$?