This is the sixth set of exercises, based on the material in Chapter III, as well as material in Chapter V. Here we will investigate isomorphisms and automorphisms of elliptic curves in characteristic 2.

Let E be an elliptic curve defined over a field K such that $\mathrm{char}(K) = 2$. Let E be defined by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where $a_i \in K$.

(1) Show that either a_1 or a_3 is different from 0 (Hint: use the differential ω). Compute $\Delta(E)$ and $j(E)$, and show that $j(E) \neq 0$ if and only if $a_1 \neq 0$.

In the rest of this exercise, we assume that $j(E) \neq 0$.

(2) Show that there exists a change of variables such that E has Weierstrass equation

$$y^2 + xy = x^3 + a_2x^2 + a_6.$$

Let E' be another elliptic curve defined over K, with Weierstrass equation

$$y'^2 + x'y' = x'^3 + a_2'x'^2 + a_6'.$$

(3) Show that any isomorphism $E' \to E$ is given by $x \mapsto x'$, $y \mapsto y' + sx'$, where $s \in \overline{K}$ is an element satisfying the quadratic equation

$$s^2 + s = a_2' - a_2.$$

(4) Show that $\mathrm{Aut}(E) \cong \mathbb{Z}/2$.

In the rest of this exercise, we assume that $K = \mathbb{F}_2$.

(5) Show that, up to isomorphism (over $\mathbb{F}_2$), there are only two elliptic curves with $j \neq 0$ defined over $\mathbb{F}_2$.

(6) Compute $E_i(\mathbb{F}_2)$, for $i = 1, 2$ in the two cases in (5).