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GEOMETRY ON GRASSMANNIANS AND APPLICATIONS TO SPLITTING BUNDLES AND SMOOTHING CYCLES

by Steven L. KLEIMAN (1)

INTRODUCTION

Let k be an algebraically closed field, V a smooth d-dimensional subscheme of projective space over k, and consider the following problems:

Problem 1 (splitting bundles). — Given a (vector) bundle G on V, find a monoidal transformation $f: \mathrm{V}' \to \mathrm{V}$ with smooth center σ such that $f^*\mathrm{G}$ contains a line bundle P.

Problem 2 (smoothing cycles). — Given a cycle Z on V, deform Z by rational equivalence into the difference $\mathrm{Z}_1 - \mathrm{Z}_2$ of two effective cycles whose prime components are all smooth.

Strengthened form. — Given any finite number of irreducible subschemes V_i of V, choose σ (resp. Z_1, Z_2) such that for all i, the intersection $\mathrm{V}_i \cap \sigma$ (resp. $\mathrm{V}_i \cap \mathrm{Z}_j$) is proper, and smooth if V_i is.

Problem 1 was first discussed by Atiyah-Serre (cf. [1], Part I, § 5) who proved that if $\mathrm{rank}(\mathrm{G}) > d = \dim(\mathrm{V})$, then already G contains a line bundle (i.e., the case: $\mathrm{F} = \mathrm{id}$, $\sigma = \emptyset$). Next, Schwarzenberger [6] proved that if V is a surface, then G is split by a sequence of monoidal transformations with smooth centers. Hironaka ([4], Corollary 2, p. 145) generalized Schwarzenberger's result to arbitrary dimension, but required $\mathrm{char}(k) = 0$.

The strengthened Problem 1 is solved below as follows. We may replace G by any Serre twist $\mathrm{G}(p)$. So we may assume G is generated by a finite number of global sections a_i. With $\mathrm{E} = \sum_i k a_i$, there exists a surjection $\mathrm{E}_\mathrm{V} = \mathrm{E} \otimes_k \mathrm{O}_\mathrm{V} \to \mathrm{G}$, and G may be viewed as a family parametrized by V of rank n quotients $\mathrm{G}(y)$ $(y \in \mathrm{V})$ of E. Hence, there corresponds a map g of V into the grassmannian $\mathrm{Grass}_n(\mathrm{E})$ of n-quotients of E such that G is the pull-back $g^*\mathrm{Q}$ of the universal quotient Q.

Problem 1 now divides in two: Let $a \in \mathrm{E}$ be a fixed section of Q, and σ the scheme (Schubert cycle) of zeros of a.

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Part I. — Let $f: Z \to \mathrm{Grass}_n(E)$ be the monoidal transformation with center σ, and find a 1-subbundle P of f^*Q.

Part II. — Assume $G(-1)$ is generated by its sections, and prove g is an embedding and $V \cap \sigma$ is appropriate if σ is sufficiently general.

Thus, Part I is Problem 1 for grassmannians, and Part II induces its solution in V.

Problem 2, posed by Borel-Haefliger, is only partially solved and possibly false in full generality. Hironaka [5] smoothed Z when $\dim(Z) \leq \min(3, (d-1)/2)$ and $\mathrm{char}(k)=0$; in any characteristic, he smoothed some multiple of Z when $\dim(Z) \leq (d-1)/2$. Below, it is proved that if $p = \mathrm{codim}(Z)$, then $(p-1)!Z$ may be strongly smoothed whenever $\dim(Z) < (d+2)/2$; in particular, 2-cycles on 4-folds and 3-cycles on 5-folds are smoothed (without torsion).

The divisorial case is typical here. If $G = O_V(Z)(n)$ with n so large that $G(-1)$ is generated by its sections, then G is very ample. Hence, Z is rationally equivalent to $Z_1 - nZ_2$ where the Z_i are hyperplane sections for suitable projective embeddings, and Bertini's theorem completes the strong smoothing.

For higher codimensions p, a bundle G is constructed using syzygies, whose p-th Chern class $c_p(G)$ is the rational equivalence class of $(p-1)!Z + nL$ for some multiple nL of a linear space section. Moreover, $G(-1)$ is generated by its sections, so that G gives rise to an embedding g of V in a grassmannian X such that $c_p(G) = g^* c_p(Q)$ with Q the universal quotient. Finally, $c_p(Q)$ is represented by a Schubert cycle σ whose singular locus has codimension $2p+2$ in X, and g is sufficiently " twisted " that $V \cap \sigma$ be as required if σ is suitably chosen.

Thus, both problems lead to grassmannians via their universal property (or functor of points). To fix notation and ideas, the theory is developed after Grothendieck [3].

The common heart of both solutions is the generalization of Bertini's smoothness theorem to sections of twisted subschemes of grassmannians by special Schubert cycles (1). The remaining Bertini theorems (on geometric irreducibility, integrality, normality, etc.) are easily verified by the same method, which is to prepare convenient local charts and to extend the usual proofs (cf. EGA V). These proofs were pioneered in abstract algebraic geometry by Zariski, to whom this article is respectfully dedicated.

§ 1. Grassmannians.

(1.1) Let S be a ground scheme and E an *r-bundle* on S (i.e., E is a locally free O_S-module of rank r). Consider the following contravariant functor X from S-schemes T to sets: Let X(T) be the set of r-quotients (bundles) G of E_T, where E_T is the pull-back of E to T. (Equivalently, X(T) is the set of $(r-n)$-subbundles A of E_T.) Clearly,

(1) Griffiths in a course at Princeton University has independently asserted a Bertini smoothness theorem for the highest Chern class.

X is a *Zariski sheaf*; namely, for all S-schemes T and for all (Zariski) open coverings $\{T_\alpha\}$ of T, the following sequence is exact:

$$X(T) \to \prod_\alpha X(T_\alpha) \rightrightarrows \prod_{\alpha,\beta} X(T_\alpha \cap T_\beta);$$

i.e., the presheaf of sets $U \mapsto X(U)$, where U is a (Zariski) open set of T, is a sheaf.

Proposition (**1.2**). — *The Grassmann functor* X *is* represented *by an* S-*scheme, denoted* $\text{Grass}_n(E)$, *together with a universal n-quotient* Q *of* $E_{\text{Grass}_n(E)}$; i.e., *for all* S-*schemes* T, *the morphisms* $f: T \to \text{Grass}_n(E)$ *and the n-quotients* G *of* E_T *are in bijective correspondence by* $G = f^*Q$.

Indeed, since the question is local on S, we may assume E is trivial (i.e., free). Let $\{e_i\}$ be an O_S-basis for E. A set J of n indices i defines a decomposition $E = E' \oplus E''$ where E' (resp. E'') is a subbundle of rank n (resp. $r-n$); thence, J defines a subfunctor X_J of X: For an S-scheme T, let $X_J(T)$ be the set of quotients G of E such that the induced map $E'_T \to G$ is surjective.

The subfunctor X_J is *open*; i.e., for any S-scheme T, identified with its functor of points Hom(—, T), and for any morphism of functors $f: T \to X$, the fiber product $T_J = X_J \times_X T$ is (represented by) an open subscheme of T. Indeed, let f correspond to a quotient G of E_T. Then T_J may be identified with the set of $t \in T$ at which the map $E'_T \to G$ is surjective; hence, T_J is open.

Since the map $E'_T \to G$ is surjective if and only if it is bijective, $X_J(T)$ may be identified with the set of maps $E_T \to E'_T$ such that the induced map $E'_T \to E'_T$ is the identity; thence, with $\text{Hom}_{O_T}(E''_T, E'_T)$. Explicitly, the canonical map $u: E_T \to G$ becomes the homomorphism $v: E''_T \to E'_T$ given by

(**1.2.1**) $$v = (u \mid E'_T)^{-1} . (u \mid E''_T).$$

Thus, X_J is represented by $\mathbf{A}_S^{r(p-n)}$.

Finally, as J runs through all sets of n indices, the X_J form a *covering* of X; namely, for all S-schemes T, we have $T = \bigcup_J T_J$. Therefore, the conclusion results from the following lemma.

Lemma (**1.3**) (Grothendieck). — *Let* X *be a contravariant functor from schemes to sets. Suppose*:

(i) X *is a Zariski sheaf.*

(ii) X *is covered by representable, open subfunctors* X_α.

Then X *is representable.*

Indeed, let $X_{\alpha\beta} = X_\alpha \times_X X_\beta$. By (ii), the map $X_{\alpha\beta} \to X_\alpha$ is an open immersion of schemes, and the X_α patch along the $X_{\alpha\beta}$ to form a scheme, which, by (i), represents X.

Remark (**1.4**). — (i) The natural correspondence between n-quotients and $(r-n)$-subbundles of E induces an isomorphism $\text{Grass}_n(E) \simeq \text{Grass}_{r-n}(E^*)$, where E^* is the dual of E.

(ii) $\mathrm{Grass}_1(\mathrm{E})$ is simply $\mathbf{P}(\mathrm{E})=\mathrm{Proj}(\mathrm{Sym}(\mathrm{E}))$, with $\mathrm{O}_{\mathbf{P}(\mathrm{E})}(1)$ as universal quotient (EGA II, (4.2.5)).

Proposition (**1.5**). — *The* Plücker morphism $\pi:\mathrm{Grass}_n(\mathrm{E})\to\mathbf{P}(\overset{n}{\Lambda}\mathrm{E})$, *defined on* T-*points by mapping an n-quotient* G *of* E *to the* 1-*quotient* $\overset{n}{\Lambda}\mathrm{G}$ *of* $\overset{n}{\Lambda}\mathrm{E}$, *is a closed immersion.*

Indeed, the question being local on S, we may proceed under the conditions of the proof of (1.2). The decomposition $\mathrm{E}=\mathrm{E}'\oplus\mathrm{E}''$ defines a decomposition $\overset{n}{\Lambda}\mathrm{E}=\overset{n}{\Lambda}\mathrm{E}'\oplus\mathrm{F}$ where $\mathrm{F}=\bigoplus_{i=1}^{n}(\overset{n-i}{\Lambda}\mathrm{E}'\oplus\overset{i}{\Lambda}\mathrm{E}'')$. Let $\mathrm{P_J}(\mathrm{T})$ be the set of 1-quotients L of $\overset{n}{\Lambda}\mathrm{E_T}$ such that the induced map $\overset{n}{\Lambda}\mathrm{E}'_\mathrm{T}\to\mathrm{L}$ is surjective. The subfunctors $\mathrm{P_J}$ form an open covering of $\mathbf{P}(\overset{n}{\Lambda}\mathrm{E})$.

Let G be an n-quotient of $\mathrm{E_T}$. The map $\mathrm{E}'_\mathrm{T}\to\mathrm{G}$ is surjective if and only if $\overset{n}{\Lambda}\mathrm{E}'_\mathrm{T}\to\overset{n}{\Lambda}\mathrm{G}$ is; hence, $\mathrm{F_J}=\pi^{-1}\mathrm{P_J}$, and it is sufficient to prove $\pi:\mathrm{F_J}\to\mathrm{P_J}$ is (represented by) a closed immersion.

Now, $\mathrm{P_J}(\mathrm{T})$ may be identified with $\mathrm{Hom}(\mathrm{F_T},\overset{n}{\Lambda}\mathrm{E}'_\mathrm{T})$, which equals $\prod_{i=1}^{n}\mathrm{Hom}(\overset{i}{\Lambda}\mathrm{E}''_\mathrm{T},\mathrm{H}_i)$ where $\mathrm{H}_i=\mathrm{Hom}(\overset{n-i}{\Lambda}\mathrm{E}'_\mathrm{T},\overset{n}{\Lambda}\mathrm{E}'_\mathrm{T})$. Since the pairing $\overset{i}{\Lambda}\mathrm{E}'_\mathrm{T}\times\overset{n-i}{\Lambda}\mathrm{E}'_\mathrm{T}\to\overset{n}{\Lambda}\mathrm{E}'_\mathrm{T}$ is nonsingular, $\mathrm{H}_i=\overset{i}{\Lambda}\mathrm{E}'_\mathrm{T}$. Thus, $\pi(\mathrm{T}):\mathrm{F_J}(\mathrm{T})\to\mathrm{P_J}(\mathrm{T})$ becomes

$$\mathrm{Hom}(\mathrm{E}''_\mathrm{T},\mathrm{E}'_\mathrm{T})\to\mathrm{Hom}(\mathrm{E}''_\mathrm{T},\mathrm{E}'_\mathrm{T})\times\prod_{i=2}^{n}\mathrm{Hom}(\overset{i}{\Lambda}\mathrm{E}''_\mathrm{T},\overset{i}{\Lambda}\mathrm{E}'_\mathrm{T})$$

$$\varphi\mapsto\varphi\times\prod_{i=2}^{n}\overset{i}{\Lambda}\varphi;$$

in this form, π is the graph morphism of a morphism of affine S-schemes. Therefore, π is a closed immersion.

The proofs of (1.2) and (1.4) also establish the following proposition. Its last assertion results in view of (EGA I, (2.2.4)).

Proposition (**1.6**). — *Let* E *be the trivial r-bundle on* S. *Then a basis of* E *defines a covering of* $\mathrm{Grass}_n(\mathrm{E})$ *by open subschemes* U *isomorphic to* $\mathbf{A}_\mathrm{S}^{n(r-n)}$. *Eeach* U *corresponds to one of the* $\binom{r}{n}$ *decompositions* $\mathrm{E}=\mathrm{E}'\oplus\mathrm{E}''$ *with* $\mathrm{rank}(\mathrm{E}')=n$, *defined by the basis. Under the Plücker morphism* π, U *is the pre-image of the complement of the hyperplane in* $\mathbf{P}(\overset{n}{\Lambda}\mathrm{E})$ *defined by* $\overset{n}{\Lambda}\mathrm{E}'$. *The set* U(T) *of* T-*points may be identified with* $\mathrm{Hom}_{\mathrm{O_T}}(\mathrm{E}''_\mathrm{T},\mathrm{E}'_\mathrm{T})$, *which if* S *is affine, is simply the set of* $n\times(r-n)$ *matrices* (t_{ij}) *with* $t_{ij}\in\Gamma(\mathrm{T},\mathrm{O_T})$.

(**1.7**) For $i=1,2$, let E_i be an r_i-bundle, $f_i:\mathrm{T}\to\mathrm{Grass}_{n_i}(\mathrm{E}_i)$ a morphism, and G_i the corresponding quotient of $\mathrm{E}_{i,\mathrm{T}}$. The quotient $\mathrm{G}_1\otimes\mathrm{G}_2$ of $(\mathrm{E}_1\otimes\mathrm{E}_2)_\mathrm{T}$ corresponds to a morphism

$$f_1\otimes f_2:\mathrm{T}\to\mathrm{Grass}_{n_1n_2}(\mathrm{E}_1\otimes\mathrm{E}_2),$$

called the *Segre product*. In particular, the *Segre morphism* is defined as

$$s=p_1\otimes p_2:\mathrm{Grass}_{n_1}(\mathrm{E}_1)\times\mathrm{Grass}_{n_2}(\mathrm{E}_2)\to\mathrm{Grass}_{n_1n_2}(\mathrm{E}_1\otimes\mathrm{E}_2)$$

where p_1, p_2 are the projections. Moreover, it is easily seen that

(1.7.1) $$f_1 \otimes f_2 = s.(f_1, f_2).$$

To undertake a local analysis, suppose the E_i are free. Let $E_i = E_i' \oplus E_i''$ be decompositions with $\mathrm{rank}(E_i') = n_i$, and consider the decomposition $E_1 \otimes E_2 = (E_1' \otimes E_2') \oplus F$ where $F = (E_1' \otimes E_2'') \oplus (E_1'' \otimes E_2') \oplus (E_1'' \otimes E_2'')$. Let U_i (resp. U) be the corresponding affine open subsets of $\mathrm{Grass}_{n_i}(E_i)$ (resp. $\mathrm{Grass}_{n_1 n_2}(E_1 \otimes E_2)$). For quotients G_i of $E_{i,T}$, the induced maps $E'_{i,T} \to G_i$ are surjective if and only if the map $(E_1' \otimes E_2')_T \to G_1 \otimes G_2$ is. Hence, $U_1 \times U_2 = s^{-1}(U)$. Identifying the set $U_i(T)$ of T-points x_i with $\mathrm{Hom}(E''_{i,T}, E'_{i,T})$ (resp. U(T) with $\mathrm{Hom}(F_T, (E_1' \otimes E_2')_T)$), we have

(1.7.2) $$s(x \times y) = 1 \otimes y + x \otimes 1 + x \otimes y.$$

Therefore, s is a closed immersion. Finally, in view of (1.7.1), if one of the morphisms $f_i : T \to \mathrm{Grass}_{n_i}(E_i)$ is an immersion, so is $f_1 \otimes f_2$.

2. Special Schubert cycles.

(2.1) Let S be a ground scheme, E an r-bundle on S, and A an a-subbundle of E (i.e., A is locally a direct summand of rank a). For a nonzero integer p satisfying $\max(0, a-n) \leq p \leq 1 + \min(a, r-n)$, consider the subfunctor $\sigma_p(A) = \sigma_p(A, E)$ of $X = \mathrm{Grass}_n(E)$ whose set of T-points consists of those n-quotients G of E_T such that the induced map $\overset{a-p+1}{\Lambda} A_T \to \overset{a-p+1}{\Lambda} G$ is 0. (Intuitively, this condition requires that $A_T \cap \mathrm{Ker}(E_T \to G)$ have $\mathrm{rank} \geq p$.)

Proposition **(2.2)**. — *The functor* $\sigma_p(A)$ *is (represented by) a closed subscheme of* $\mathrm{Grass}_n(E)$ *(called the p-th special Schubert cycle defined by* A*).*

Indeed, $\sigma_p(A)$ is a Zariski sheaf. So, we may work locally on S and on $\mathrm{Grass}_n(E)$. Suppose then that S is affine, that A (resp. E) is trivial, and that A is a direct summand of E. Let $E = E' \oplus E''$ be a decomposition with E′ trivial of rank n, let U be the corresponding open affine subscheme of $\mathrm{Grass}_n(E)$, and identify U(T) with $\mathrm{Hom}(E''_T, E'_T)$. Then $(\sigma_p(A) \cap U)(T)$ becomes the set of $t \in \mathrm{Hom}(E''_T, E'_T)$ such that

(2.2.1) $$(\overset{a-p+1}{\Lambda}(1+t))(\overset{a-p+1}{\Lambda} A_T) = 0,$$

a condition which may be expressed in terms of polynomials; whence, the assertion.

Proposition **(2.3)**. — *Suppose* S *is affine, and* E *is a trivial bundle with* O_S*-basis* $e_1, \ldots, e_r$. *Let* U *be the open affine subscheme of* $\mathrm{Grass}_n(E)$ *corresponding to the basis elements* $e_1, \ldots, e_n$, *and identify* U(T) *with the set of* $n \times (r-n)$ *matrices* $t = (t_{ij})$ *with* $t_{ij} \in \Gamma(T, O_T)$. *If* $A = O_S f$ *where* $f = \sum_{j=1}^{r} s_j e_j$ *with* $s_j \in \Gamma(S, O_S)$, *then* $(\sigma_1(A) \cap U)(T)$ *is cut out of* U(T) *by the following set of linear equations*:

$$s_i + \sum_{j=1}^{r-n} s_{j+n} t_{ij} = 0, \qquad i = 1, \ldots, n.$$

Indeed, (2.2.1) becomes $\sum_{i=1}^{n} s_i e_i + \sum_{i=n+1}^{r} s_i \sum_{j=1}^{r} t_{ji} e_i = 0$; whence the conclusion.

Remark (**2.4**). — (i) Let $p_0 = \max(0, a-n)$ and $p_1 = 1 + \min(a, r-n)$. Then,

$$\sigma_{p_0}(A) = \mathrm{Grass}_n(E) \supset \sigma_{p_0+1}(A) \supset \ldots \supset \sigma_{p_1}(A) = \emptyset.$$

(ii) $\sigma_a(A) = \mathrm{Grass}_n(E/A)$, and on T-points, its inclusion morphism into $\mathrm{Grass}_n(E)$ becomes the canonical map taking an n-quotient of E/A to an n-quotient of E.

(iii) $\sigma_1(A)$ may be interpreted as the scheme of zeros of the module A of " sections " of the universal quotient Q on $X = \mathrm{Grass}_n(E)$, and for any morphism $t: T \to X$, $t^{-1}\sigma_1(A)$ as the scheme of zeros of the module A_T of " sections " of Q_T.

(iv) Suppose $a = n$, and embed $\mathrm{Grass}_n(E)$ in $\mathbf{P}(\overset{n}{\Lambda} E)$ via the Plücker immersion. Then $\sigma_1(A)$ is the hyperplane section that $\overset{n}{\Lambda} A$ defines.

(v) For any base change $R \to S$, $\sigma_p(A_R) = \sigma_p(A) \times_S R \hookrightarrow \mathrm{Grass}_n(E_R) = \mathrm{Grass}_n(E) \times_S R$.

Lemma (**2.5**). — *Let* R *be a ring,* E *an* R*-module,* A *and* B *submodules of* E, *and* $C = A + B$. *Assume that* B *is a direct summand of* E *and that* B *is a free* R*-module. Then for an integer* $q \geq 0$, *the following three conditions are equivalent*:

(i) *The canonical map* $\overset{q}{\Lambda} A \to \overset{q}{\Lambda}(E/B)$ *is* 0.

(ii) *The canonical map* $\overset{q}{\Lambda}(C/B) \to \overset{q}{\Lambda}(E/B)$ *is* 0.

(iii) *The canonical map* $\overset{q+b}{\Lambda} C \to \overset{q+b}{\Lambda} E$ *is* 0, *where* $b = \mathrm{rank}(E)$.

Indeed, since the canonical map $A \to C/B$ is surjective, its exterior power $\overset{q}{\Lambda} A \to \overset{q}{\Lambda}(C/B)$ is surjective; it follows that (i) is equivalent to (ii). Let $E = B \oplus G$ and $D = C \cap G$; whence, $C = B \oplus D$. Then (ii) becomes the condition that $\overset{q}{\Lambda} D \to \overset{q}{\Lambda} G$ is 0, or equivalently that $\overset{q+i}{\Lambda} D \to \overset{q+i}{\Lambda} G$ is 0 for all $i \geq 0$. Since

$$\overset{q+b}{\Lambda} C = \bigoplus_{0 \leq j \leq b} (\overset{j}{\Lambda} B \otimes \overset{q+b-j}{\Lambda} D) \qquad \text{and} \qquad \overset{q+b}{\Lambda} E = \bigoplus_{0 \leq j \leq b} (\overset{j}{\Lambda} B \otimes \overset{q+b-j}{\Lambda} G),$$

it follows that (ii) is equivalent to (iii).

Proposition (**2.6**). — *Let* t *be a* T*-point of* $\mathrm{Grass}_n(E)$, *and* B *the corresponding* $(r-n)$*-subbundle of* E_T. *Then* t *lies in* $\sigma_p(A) - \sigma_{p+1}(A)$ *if and only if the sheaf-theoretic sum* $A_T + B$ *is an* $(a + (r-n) - p)$*-subbundle.*

Indeed, the following statements are equivalent:

(i) The point t lies in $\mathrm{Grass}_n(E) - \sigma_{p+1}(A)$.

(ii) For all geometric points $\mathrm{Spec}(k) \to T$, the map $\overset{a-p}{\Lambda} A_k \to \overset{a-p}{\Lambda}(E_k/B_k)$ is nonzero.

(iii) $\dim(A_k + B_k) \geq a + (r-n) - p$.

(iv) Locally on T, there exists an $(a + (r-n) - p)$-subbundle D of E_T such that sheaf-theoretically $D \subset A_T + B_T$.

Applying the equivalence (2.5) (i)$\Leftrightarrow$(iii) locally on T, we find the equivalence of the following statements, and thus establish the assertion:

a) The point t lies in $\sigma_p(\mathrm{A})-\sigma_{p+1}(\mathrm{A})$.

b) The map $\overset{a-p+1}{\Lambda}\mathrm{A}_\mathrm{T}\to\overset{a-p+1}{\Lambda}(\mathrm{E}_\mathrm{T}/\mathrm{B})$ is 0, and (i) holds.

c) The map $\overset{q}{\Lambda}(\mathrm{A}_\mathrm{T}+\mathrm{B})\to\overset{q}{\Lambda}\mathrm{E}_\mathrm{T}$ is 0 with $q=a+r-n-p+1$, and (iv) holds.

d) The map $\mathrm{A}_\mathrm{T}+\mathrm{B}\to\mathrm{E}_\mathrm{T}/\mathrm{D}$ is 0.

(2.7) Let $\mathrm{F}=\mathrm{E}/\mathrm{A}$, $\mathrm{Y}=\mathrm{Grass}_{n-a+p}(\mathrm{F})$, and R be the universal quotient on Y. Define an $(r-n+a-p)$-bundle K on Y via the following diagram:

(2.7.1)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{A}_\mathrm{Y} & \longrightarrow & \mathrm{K} & \longrightarrow & \mathrm{J} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{A}_\mathrm{Y} & \longrightarrow & \mathrm{E}_\mathrm{Y} & \longrightarrow & \mathrm{F}_\mathrm{Y} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathrm{R} & \xrightarrow{\mathrm{id}} & \mathrm{R} & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Theorem **(2.8)**. — *Under the conditions of* (2.7), *there exists a canonical morphism f yielding a diagram*

$$\begin{array}{ccccc}
\mathrm{Grass}_{a-p}(\mathrm{K})=\sigma_p(\mathrm{A}_\mathrm{Y},\mathrm{K}) & & \hookleftarrow & & [\sigma_p(\mathrm{A}_\mathrm{Y},\mathrm{K})-\sigma_{p+1}(\mathrm{A}_\mathrm{Y},\mathrm{K})]\\
\swarrow \qquad\qquad \searrow{\scriptstyle f} & & & & \searrow{\scriptstyle g\ \simeq}\\
\mathrm{Y}=\mathrm{Grass}_{n-a+p}(\mathrm{E}/\mathrm{A}) \qquad \sigma_p(\mathrm{A}) & & \hookleftarrow & & [\sigma_p(\mathrm{A})-\sigma_{p+1}(\mathrm{A})]
\end{array}$$

with cartesian square, and g the restriction of f is an isomorphism.

Indeed, a T-point t of $\mathrm{Grass}_{a-p}(\mathrm{K})$ corresponds to an $(r-n)$-subbundle B of K_T: since $\overset{a-p+1}{\Lambda}(\mathrm{K}_\mathrm{T}/\mathrm{B})$ is 0, the map $\overset{a-p+1}{\Lambda}\mathrm{A}_\mathrm{T}\to\overset{a-p+1}{\Lambda}(\mathrm{E}_\mathrm{T}/\mathrm{B})$, which factors through it, is 0. So, B corresponds to a T-point $f(t)$ of $\sigma_p(\mathrm{A})$. Thus f is defined. By (2.6), t lies in $\sigma_p(\mathrm{A}_\mathrm{Y},\mathrm{K})-\sigma_{p+1}(\mathrm{A}_\mathrm{Y},\mathrm{K})$ (resp. $f(t)$ lies in $\sigma_p(\mathrm{A})-\sigma_{p+1}(\mathrm{A})$) if and only if $\mathrm{A}_\mathrm{T}+\mathrm{B}=\mathrm{K}_\mathrm{T}$; hence, the square is cartesian.

It remains to construct an inverse h to g. Let then s be a T-point of $\sigma_p(\mathrm{A})-\sigma_{p+1}(\mathrm{A})$. It corresponds to an $(r-n)$-subbundle B of E_T such that $\mathrm{A}_\mathrm{T}+\mathrm{B}$ is an $(a+r-n-p)$-bundle by (2.6). So, there exists a unique morphism $\mathrm{T}\to\mathrm{Y}$ such that $\mathrm{A}_\mathrm{T}+\mathrm{B}=\mathrm{K}_\mathrm{T}$. The subbundle B of K_T then defines a morphism $h(s):\mathrm{T}\to\mathrm{Grass}_{a-p}(\mathrm{K})$, completing the proof.

Corollary **(2.9)**. — (i) $\sigma_p(\mathrm{A})-\sigma_{p+1}(\mathrm{A})$ *is smooth over* S, *of relative dimension* $n(r-n)-(n-a+p)p$.

(ii) *If* $p\geq 1$, *then* $\sigma_p(\mathrm{A})$ *is not smooth at any point of* $\sigma_{p+1}(\mathrm{A})$.

Indeed, (i) results in view of (1.6). Hence the codimension of

$$f^{-1}(\sigma_{p+1}(\mathrm{A}))=\sigma_{p+1}(\mathrm{A_Y}, \mathrm{K})$$

in $\mathrm{Z}=\mathrm{Grass}_{a-p}(\mathrm{K})$ is $p+1$, and it follows that (ii) holds. For it suffices to consider the geometric fibers over s and to apply the fact that, for a birational morphism $f:\mathrm{X}\to\mathrm{Y}$ with Y factorial, the exceptional locus is of codimension one in X.

Corollary **(2.10)**. — *Suppose* $r-n\leq a$. *Then* $\sigma_{r-n}(\mathrm{A})=\mathrm{Grass}_{a+n-r}(\mathrm{A})$, *and on* T-*points its inclusion morphism into* $\mathrm{Grass}_n(\mathrm{E})$ *becomes the canonical map taking an* $(r-n)$-*subbundle of* $\mathrm{A_T}$ *to one of* $\mathrm{E_T}$.

In particular, if $r-n=1$, *then* $\mathrm{Grass}_n(\mathrm{E})=\mathbf{P}(\mathrm{E}^*)$ *contains* $\sigma_1(\mathrm{A})=\mathbf{P}(\mathrm{A}^*)$ *as a linear subspace.*

Remark **(2.11)**. — Let A (resp. B) be the universal subbundle on $\mathrm{Grass}_m(\mathrm{E})$ (resp. on $\mathrm{Grass}_n(\mathrm{E})$):

(i) $\sigma_p(\mathrm{A})$ defines an algebraic family of special Schubert cycles on $\mathrm{Grass}_n(\mathrm{E})$ parametrized by $\mathrm{Grass}_m(\mathrm{E})$, and this family is universal in an obvious sense.

(ii) The canonical isomorphism $\mathrm{Grass}_n(\mathrm{E})\times\mathrm{Grass}_m(\mathrm{E})\simeq\mathrm{Grass}_m(\mathrm{E})\times\mathrm{Grass}_n(\mathrm{E})$ carries $\sigma_p(\mathrm{A})$ isomorphically onto $\sigma_p(\mathrm{B})$.

Indeed, a T-point t of the product lies in $\sigma_p(\mathrm{A})$ (resp. in $\sigma_p(\mathrm{B})$) if and only if the map $\overset{q}{\Lambda}(\mathrm{A_T}+\mathrm{B_T})\to\overset{q}{\Lambda}\mathrm{E_T}$ is 0 with $q=(r-n)+(r-m)-p+1$ in view of the equivalence (2.5) (i)$\Leftrightarrow$(iii) applied locally on T.

(iii) Suppose $r-m\leq r-n$. Then $\sigma_{r-m}(\mathrm{A})=\sigma_{r-m}(\mathrm{B})$ is called the *incidence correspondence*, a T-point of $\mathrm{Grass}_n(\mathrm{E})\times\mathrm{Grass}_m(\mathrm{E})$ lies in it if and only if $\mathrm{A_T}\subset\mathrm{B_T}$. It is the bundle of grassmannians $\mathrm{Grass}_n(\mathrm{E_X}/\mathrm{A})$ over $\mathrm{X}=\mathrm{Grass}_n(\mathrm{E})$, and $\mathrm{Grass}_{r-m}(\mathrm{B}^*)$ over $\mathrm{Grass}_{r-m}(\mathrm{E}^*)=\mathrm{Grass}_m(\mathrm{E})$.

Proposition **(2.12)**. — *Under the conditions of* (2.8) *with* $p=0$, *consider the following diagram with canonical morphisms:*

$$\begin{array}{ccc} & \mathrm{Grass}_m(\mathrm{F})\times[\mathrm{Grass}_n(\mathrm{E})-\sigma_1(\mathrm{A})] & \\ \quad{\swarrow}^{g} & & {\searrow}^{h}\quad \\ \mathrm{Grass}_m(\mathrm{E})\times\mathrm{Grass}_n(\mathrm{E}) & & \mathrm{Grass}_m(\mathrm{F})\times\mathrm{Grass}_{n-a}(\mathrm{F}). \end{array}$$

Let B (resp. J) *be the universal subbundle on* $\mathrm{Grass}_n(\mathrm{E})$ (resp. *on* $\mathrm{Grass}_{n-a}(\mathrm{F})$). *Then* $g^{-1}\sigma_p(\mathrm{B})=h^{-1}\sigma_p(\mathrm{J})$.

Indeed, let P be the universal quotient on $\mathrm{Grass}_m(\mathrm{F})$. For any T-point t of $\mathrm{Grass}_m(\mathrm{F})\times(\mathrm{Grass}_n(\mathrm{E})-\sigma_1(\mathrm{A}))$, $g(t)$ lies on $\sigma_p(\mathrm{B})$ if and only if the map $\overset{q}{\Lambda}\mathrm{B_T}\to\overset{q}{\Lambda}\mathrm{P_T}$

is o with $q=(r-n)-p+1$, and $h(t)$ lies on $\sigma_p(\mathrm{J})$ if and only if the map $\overset{q}{\Lambda}\mathrm{J}_\mathrm{T}\to\overset{q}{\Lambda}\mathrm{P}_\mathrm{T}$ is o with $q=(r-a)-(n-a)-p+1$. Since $\mathrm{pr}_2(t)$ lies off $\sigma_1(\mathrm{A})$, the map $\mathrm{B}_\mathrm{T}\to\mathrm{J}_\mathrm{T}$ is surjective; whence, the conclusion.

Corollary **(2.13)**. — *Preserve the conditions of* (2.12). *Then:*

(i) *With respect to the inclusion* $\mathrm{Grass}_m(\mathrm{F})\hookrightarrow\mathrm{Grass}_m(\mathrm{E})$, *a* $\sigma_p(\mathrm{B}_t)$ *on* $\mathrm{Grass}_m(\mathrm{E})$ *will in general (precisely when* $t\notin\sigma_1(\mathrm{A})$*) induce a* $\sigma_p(\mathrm{J}_t)$ *on* $\mathrm{Grass}_m(\mathrm{F})$, *and every* $\sigma_p(\mathrm{J}_t)$ *is obtained in this way; in fact, this correspondence is simply the natural morphism* $\mathrm{Grass}_n(\mathrm{E})-\sigma_1(\mathrm{A})\to\mathrm{Grass}_{n-a}(\mathrm{F})$.

(ii) *Let* B′ (resp. J′) *be the universal subbundle on* $\mathrm{Grass}_m(\mathrm{E})$ (resp. *on* $\mathrm{Grass}_m(\mathrm{F})$). *With respect to the morphism* $\mathrm{Grass}_n(\mathrm{E})-\sigma_1(\mathrm{A})\to\mathrm{Grass}_{n-a}(\mathrm{F})$, *the inverse image of a* $\sigma_p(\mathrm{J}'_t)$ *on* $\mathrm{Grass}_{n-a}(\mathrm{F})$ *is a uniquely determined* $\sigma_p(\mathrm{B}'_t)$ *on* $\mathrm{Grass}_n(\mathrm{E})$, *restricted to the open set; in fact, this correspondence is simply the inclusion* $\mathrm{Grass}_m(\mathrm{F})\hookrightarrow\mathrm{Grass}_m(\mathrm{E})$.

Proposition **(2.14)**. — *Under the conditions of* (2.3), *let* A *be generated by*

$$e_1, \dots, e_{a-p}, f_1, \dots, f_p$$

where $f_i=\sum_{j=1}^{r} s_j^{(i)}e_j$ *with* $s^{(i)}\in\Gamma(\mathrm{S},\mathrm{O}_\mathrm{S})$. *Then* $(\sigma_p(\mathrm{A})\cap\mathrm{U})(\mathrm{T})$ *is cut out by the linear equations:*

$$s_i^{(l)}+\sum_{j=1}^{r-n} s_{j+n}^{(l)}t_{ij}=0 \qquad (l=1,\dots,p \text{ and } i=a-p+1,\dots,n).$$

Indeed, let F be the bundle generated by $e_q, \dots, e_r$ with $q=a-p+1$, and U′ the affine subscheme of $\mathrm{X}'=\mathrm{Grass}_{n-a+p}(\mathrm{F})$ corresponding to $e_q, \dots, e_n$. The natural morphism $\mathrm{Grass}_n(\mathrm{E})-\sigma_1(\mathrm{A}')\to\mathrm{X}'$ where A′ is generated by $e_1, \dots, e_{q-1}$ is easily seen to become the projection $\mathrm{U}(\mathrm{T})\to\mathrm{U}'(\mathrm{T})$ given in coordinates by

$$\begin{pmatrix} * \\ \hline t_{q1}\cdots\cdots \\ \vdots \\ t_{n1}\cdots\cdots \end{pmatrix} \longmapsto \begin{pmatrix} t_{q1}\cdots\cdots\cdots \\ \vdots \\ \\ t_{n1}\cdots\cdots\cdots \end{pmatrix}$$

In view of (2.13) (ii), we may replace E by F, and f_i by $\sum_{j=q}^{r} s_j^{(i)}e_j$. Thus, we may assume $p=a$; whence, $\sigma_p(\mathrm{A})=\bigcap_{j=1}^{p}\sigma_1(\mathrm{O}_\mathrm{S}f_j)$, and the conclusion results from (2.3).

3. Bertini's theorem.

Fix a ground scheme S, and an r-bundle E on S.

Definition **(3.1)**. — *An embedding* $\mathrm{V}\hookrightarrow\mathrm{Grass}_n(\mathrm{E})$ *is called* twisted *if it is the Segre product of a morphism* $\mathrm{V}\to\mathrm{Grass}_n(\mathrm{E}_1)$ *and an embedding* $\mathrm{V}\hookrightarrow\mathbf{P}(\mathrm{E}_2)$.

Remark **(3.2)**. — (i) If V is a twisted subscheme of $\mathrm{Grass}_n(\mathrm{E})$, then so is any subscheme V_0 of V.

(ii) Any embedding $V \hookrightarrow \mathbf{P}(E)$ is twisted, for it is the Segre product of the structure map $V \to \mathbf{P}(O_S) = S$ and itself.

(iii) Let S be affine, V quasi-projective over S with very ample sheaf $O_X(1)$, and G an n-bundle on V. Then for $p \gg 0$, there exists a twisted embedding of V in a suitable $\mathrm{Grass}_n(E)$ such that the pull-back of the universal quotient Q is $G(p)$. In fact, p need only be so large that $G(p-1)$ is a quotient of a trivial bundle E_1 (EGA II, (4.5.5)).

Theorem **(3.3).** — *Let* S *be a noetherian scheme,* V *a twisted subscheme of* $X = \mathrm{Grass}_n(E)$ *of pure relative dimension* d (resp. *which is smooth over* S), A *the universal subbundle on* $Y = \mathrm{Grass}_{r-a}(E)$, *and* p *an integer satisfying*

$$\max(0, a-n) < p \leq \min(a, r-n).$$

Then there exists a dense open set W *of* Y *such that the* W*-scheme*

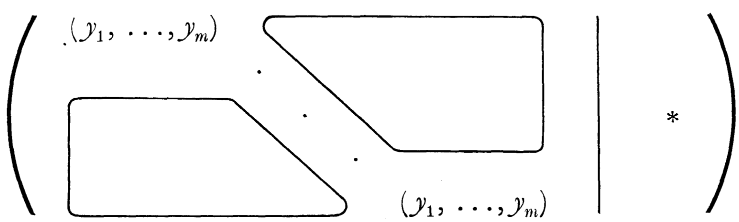
$$V_W \cap (\sigma_p(A_W, E_W) - \sigma_{p+1}(A_W, E_W))$$

has pure relative dimension $d-(n-a+p)p$ (resp. *is smooth*)*; in particular, it is empty if (and only if)* $d < (n-a+p)p$.

Indeed, in view of the constructibility of the properties in question (EGA IV, (9.5.6), (17.7.11)), it suffices to analyze the generic fibers; we assume therefore that $S = \mathrm{Spec}(k)$ with k a field, that $E = E_1 \otimes E_2$ with k-basis $e_l = e_i^{(1)} \otimes e_j^{(2)}$ suitably ordered ($l = 1, \ldots, r$), and that A_K is generated by elements $f_i = \sum_{j=1}^{r} s_{ij} e_j$ ($i = 1, \ldots, a$) with $K = k(s_{ij})$ purely transcendental.

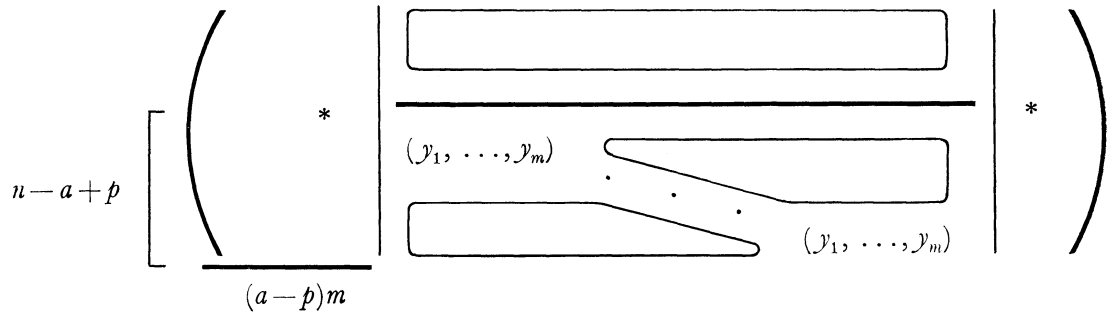
The questions being local, fix a point $y \in V_K$. Reordering the $e_j^{(i)}$ ($i = 1, 2$), we may assume y maps into the affine subscheme U_i of $\mathrm{Grass}_n(E_1)$ (resp. $\mathbf{P}(E_2)$) corresponding to $e_1^{(1)}, \ldots, e_n^{(1)}$ (resp. $e_1^{(2)}$); then y lies in the affine subscheme U_0 of $X_K = \mathrm{Grass}_n(E_K)$ corresponding to $e_1^{(1)} \otimes e_1^{(2)}, \ldots, e_n^{(1)} \otimes e_1^{(2)}$ by (1.7). If $y \notin \sigma_{p+1}(A_K)$, reordering the f_i and $e_1^{(1)}, \ldots, e_n^{(1)}$, we may assume y lies as well in the affine subscheme U of X_K corresponding to the basis parts $f_1, \ldots, f_{a-p}, e_{a-p+1}, \ldots, e_n$ and $e_{n+1}, \ldots, e_r$.

Let Ω be a universal domain for K, and let $(y_1, \ldots, y_m) \in U_2(\Omega)$ ($m = r_2 - 1$) be a coordinatized point lying on (the image of) V in $\mathbf{P}(E_2)$. In view of (1.7.2), this Ω-point of V acquires the following coordinatization in $U_0(\Omega)$:

(3.3.1) 

Suppose the point lies as well in U. In view of (1.2.1), its U-coordinatization is obtained from (3.3.1) by premultiplication with an $n \times n$ matrix M which may be computed as follows: Let u be the map $E_\Omega \to E'_\Omega$ corresponding to (3.3.1); then, M is the matrix of $(u \mid F_\Omega)^{-1}$ where F is generated by $f_1, \ldots, f_{a-p}, e_{a-p+1}, \ldots, e_n$; it follows that M has the form $\left(\begin{array}{c|c} * & 0 \\ \hline * & I \end{array}\right)$ where I is the $(n-a+p)$-identity. Therefore the U-coordinatization is:

(3.3.2)



By virtue of (2.14), $U \cap \sigma_1(A_K)$ is cut out by the linear equations

$$s'_{li} + \sum_{j=1}^{r-n} s'_{l,j+n} t_{ij} = 0 \qquad (l = a-p+1, \ldots, a \text{ and } i = a-p+1, \ldots, n)$$

where the s'_{ij} are certain, obvious, generic linear combinations of the s_{ij}.
Consider the subscheme $V_{\alpha,\beta}$ of $V_K \cap U$ cut out by the above equations for

$$(a-p+1, a-p+1) \leq (i, l) \leq (\alpha, \beta)$$

ordered lexicographically; for example, $V_{a-p,a} = V_K \cap U$. In view of (3.3.3), $V_{\alpha,\beta}$ is cut out of $V_K \cap U$ in $U \cap U_0$ by the equations

$$s_{li} + \left(\sum_{j=1}^{mp} + \sum_{j=mn+1}^{r-n}\right) s_{l,j+n} t_{ij} = 0 \qquad \text{for} \qquad (a-p+1, a-p+1) \leq (i, l) \leq (\alpha, \beta).$$

So, U being defined over $k' = k(s_{ij} \mid i \leq a-p)$, then $V_{\alpha\beta} \cap U_0$ $(\alpha > a-p)$ is defined over

$$k_{\alpha,\beta} = k_{(\alpha,\beta)-1}(s_{\beta,j} \mid j = \alpha, n+m(\alpha-1)+1, \ldots, n+m\alpha)$$

where $(\alpha, \beta)-1$ equals $(\alpha-1, a)$ if $\beta = a-p+1$ (resp. $(\alpha, \beta-1)$ otherwise) and $k_{a-p,a} = k'(s_{ij} \mid j = n+1, \ldots, n+(a-p)m, n+mn, \ldots, r)$. Moreover, in view of (3.3.2), the projection of U onto $\operatorname{Spec}(k_{\alpha\beta}[t_{\alpha,m(\alpha-1)+1}, \ldots, t_{\alpha,m\alpha}])$ embeds $V_{\alpha,\beta} \cap U_0$. Therefore, the conclusion results by induction on (α, β) from the following lemma applied with $V = V_{\alpha,\beta} \cap U_0$.

Lemma **(3.4)**. — *Let k be a field,* $A = k[t_1, \ldots, t_N]$ *a polynomial ring,* $U = \operatorname{Spec}(A)$ *affine* N*-space, and* V *an irreducible subscheme of* U *of dimension* d (≥ 0). *Let* $s, s_1, \ldots, s_m$ *be indeterminates,* $L = k(s_1, \ldots, s_m)$, *and* $K = L(s)$. *Let* $f \in A_L$, *and* $W = V_K \cap \{f = s\}$. *Then,* $\dim(W) \leq d-1$.

Let $\pi : U \to U' = \operatorname{Spec}(k[t_1, \ldots, t_m])$ *be the projection. Suppose* $\pi : V \to U'$ *is an embedding and f has the form*

$$f = s_1 t_1 + \ldots + s_m t_m + f_1 \quad \text{with } f_1 \in k[t_{m+1}, \ldots, t_N].$$

Then $\dim(W) = d-1$, *and* W *is smooth if* V *is.*

Indeed for convenience, we may assume k is algebraically closed. Let Ω be a universal domain for K. If $(y_1, \ldots, y_N) \in W(\Omega)$, then $s \in L(y_i)$; hence,

$$\text{(3.4.1)} \qquad \operatorname{tr.deg}(L(y_i)/L) \geq 1.$$

However, $V(\Omega)$ contains points (y_i) algebraic over L. Thus, $\dim(W) \leq d-1$.

Suppose the additional hypotheses hold. Let $(y_i) \in V(\Omega)$ be a generic point over L, and set $s' = f(y_i)$. Then s' lies in $L(y_i)$, which is a regular extension of L. If s' is algebraic over L, then $s' \in L$; hence, $y_1, \ldots, y_m \in k$, for $k(y_i)$ and L are linearly disjoint over k [1]. As $\pi | V$ is an embedding, it follows that $d=0$. Thus if $d \geq 1$, then s' is transcendental over L, and an L-automorphism of Ω which maps s' to s will map (y_i) to a point in $W(\Omega)$. Therefore, $\dim(W) = d-1$.

Suppose V is smooth at $(y_i) \in W(\Omega)$. Reordering $t_1, \ldots, t_m$, take $g_{d+1}, \ldots, g_N \in \mathfrak{I}_k(V)$ with full-rank jacobian $\mathcal{J} = (a_{ij} \mid d+1 \leq i, j \leq N)$ where $a_{ij} = \dfrac{\partial g_i}{\partial t_j}(y_l)$ as follows:

Take $g_{d+1}, \ldots, g_m \in k[t_1, \ldots, t_m]$ vanishing on $\pi(V)$ with full-rank jacobian $(a_{ij} \mid d+1 \leq i, j \leq m)$; they exist because $\pi(V)$ is smooth at $(y_1, \ldots, y_m)$. Take $g_{m+1}, \ldots, g_N \in \mathfrak{I}_k(V)$ with full-rank jacobian $(a_{ij} \mid m+1 \leq i, j \leq N)$; they exist because the projection of the normal space to V at (y_i) into the space of $dt_{m+1}, \ldots, dt_N$ is surjective, $\pi | V$ being an embedding.

It remains to prove that for $j = 1, \ldots, d$, the jacobian determinants

$$\begin{vmatrix} s_j & s_{d+1} \cdots s_N \\ a_{d+1,j} & \\ \vdots & \boxed{\mathcal{J}} \\ a_{N,j} & \end{vmatrix}$$

are not all 0, where $s_i = \dfrac{\partial f}{\partial t_i}(y_l)$. Suppose they are 0. Then $\operatorname{tr.deg}(L(y_i)/k(y_i)) \leq m-d$. Consider the diagram

$$\begin{array}{ccccc} & & L(y_i) & & \\ & \diagup & & \diagdown & \\ k(y_i) & & & & L \\ & \diagdown & & \diagup & \\ & & k & & \end{array}$$

In it, $\operatorname{tr.deg}(k(y_i)/k) \leq d$ because $(y_i) \in V(\Omega)$, and $\operatorname{tr.deg}(L/k) = m$ by construction. Therefore $\operatorname{tr.deg}(L(y_i)/L) = 0$, which contradicts (3.4.1).

---

[1] The elements $s', s_1, \ldots, s_m, 1 \in L$ become linearly dependent over $k(y_i)$, so there exist elements $a_1, \ldots, a_m, a \in k$ such that $s' = s_1 a_1 + \ldots + s_m a_m + a$. Since $s_1, \ldots, s_m, 1$ are linearly independent over k, $y_1 = a_1, \ldots, y_m = a_m, f_1(y_i) = a$.

Corollary **(3.5)**. — *Under the conditions of* (3.3), *suppose* $p=\min(a, r-n)$ *or* $d<(n-a+p+1)(p+1)$. *Then* $V_W \cap \sigma_p(A_W)$ *has pure relative dimension* $d-(n-a+p)p$ (resp. *is smooth*).

Corollary **(3.6)**. — *Let k be an infinite field,* V *a pure d-dimensional, algebraic k-scheme equipped with an ample sheaf* $O_V(1)$, *and* G *an n-bundle on* V. *Let* α *be a general section of* $G(p)$ *for some* $p \gg 0$. *If* $d<n$, *then* α *has no zeros; if* V *is smooth, then* α *meets the zero section transversally;* i.e., *the scheme* σ *of zeros of* α *is smooth of codimension n.*

Indeed, k being infinite, the k-points of the projective space parametrizing the sections α are dense, rendering " general " meaningful. The assertions result immediately from (3.2) (iii), (2.4) (iii) and (3.5) with $a=p=1$.

Remark **(3.7)**. — Modified slightly, the argument of (3.9) shows that for an arbitrary subscheme V of $\text{Grass}_n(E)$ of relative dimension d, there exists a dense open set W of $Y=\text{Grass}_{r-a}(E)$ such that $V_W \cap \sigma_p(A_W)$ has pure relative dimension $\leq d-(n-a+p)p$.

The inequality may however be strict. For example, suppose $E=A \oplus F$, and $r-n \geq n$ and $n-a \geq a$. Then the intersection of $V=\sigma_{r-n}(F)$ and $\sigma_a(A)$ is empty, although $d=(n-a)(r-n) \geq na$.

4. Splitting bundles.

(4.1) Employing the notation of (2.7), set $Z=\text{Grass}_{a-p}(K)$, let P be the universal quotient on Z, and define an n-bundle G on Z via the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & K_Z & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \text{id.}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & E_Z & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & R_Z & \xrightarrow{\text{id.}} & R_Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{4.1.1}
$$

Since $\overset{a-p+1}{\Lambda} P=0$, the map $\overset{a-p+1}{\Lambda} A_Z \to \overset{a-p+1}{\Lambda} Q_Z$, which factors through it, is 0. There exists therefore a canonical morphism $Z \to \sigma_p(A)$ such that $G=Q_Z$ and $L=B_Z$ where Q is the universal quotient on $X=\text{Grass}_n(E)$ and B the universal subbundle; this morphism is simply the morphism f of (2.8).

Diagram (2.7.1) now yields the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_Z & \longrightarrow & E_Z & \longrightarrow & F_Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{id.} & & \downarrow & & \\ 0 & \longrightarrow & K_Z & \longrightarrow & E_Z & \longrightarrow & R_Z & \longrightarrow & 0 \end{array}$$

It together with the last two columns of (4.1.1) gives rise to the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_Z & \longrightarrow & E_Z & \longrightarrow & F_Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_Z & \longrightarrow & Q_Z & \longrightarrow & R_Z & \longrightarrow & 0 \end{array} \tag{4.1.2}$$

which is universal in the sense specified in the next theorem.

Theorem **(4.2)**. — *Preserving the notation of* (4.1), *consider a morphism*

$$t : T \to X = \mathrm{Grass}_n(E).$$

Then a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_T & \longrightarrow & E_T & \longrightarrow & F_T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & Q_T & \longrightarrow & R_1 & \longrightarrow & 0 \end{array} \tag{4.2.1}$$

such that P_1 *is an* $(a-p)$*-subbundle of* Q_T, *uniquely defines a factorization of* t *through* $f : Z = \mathrm{Grass}_{a-p}(K) \to \sigma_p(A)$; *moreover,* $P_1 = P_T$ *and* $R_1 = R_T$, *and* (4.2.1) *is the pull-back of* (4.1.2).

Indeed, the map $F_T \to R_1$ being surjective, there exists a unique map $T \to Y = \mathrm{Grass}_{n-a+p}(F)$ such that $R_1 = R_T$. So (4.2.1) yields the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & E_T & \xrightarrow{\text{id}} & E_T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & Q_T & \longrightarrow & R_T & \longrightarrow & 0 \end{array}$$

whose exact serpent (homology) sequence reduces simply to $0 \to B_T \to K_T \to P_1 \to 0$. Consequently, there exists a unique map $t' : T \to Z$ such that $P_1 = P_T$ and $B_T = L_T$. The middle line of (4.1.1) then shows that $t^*Q = t'^*Q_Z$; hence, $T \xrightarrow{t'} Z \to X$ is indeed a factorization of t.

Corollary **(4.3)**. — *In the notation of* (4.1), *there exists a canonical, cartesian diagram of schemes over* $X = \mathrm{Grass}_n(E)$,

$$\begin{array}{ccc} \mathrm{Grass}_{n-a+p}(Q) & \hookrightarrow & \mathrm{Grass}_{n-a+p}(E_X) \\ \uparrow & \square & \uparrow \\ Z = \mathrm{Grass}_{a-p}(K) & \longrightarrow & \sigma_a(A_X) \end{array}$$

Indeed, the assertion is but a reformulation of (4.2).

Theorem **(4.4)**. — *Preserving the notation of* (4.1), *take* $p=0$, *let* M *be the* n*-bundle* $(\overset{a}{\Lambda} Q)^* \otimes \overset{a}{\Lambda} A_X$ *on* $X = \mathrm{Grass}_n(E)$, *and let* I *be the natural image of* M *in* O_X, *which is the ideal of* $\sigma_1(A)$. *Then the morphism* $f: Z = \mathrm{Grass}_a(K) \to X$ *is the birational blowing-up of* I, i.e., *the monoidal transformation with center* $\sigma_1(A)$.

Indeed, for convenience, we may assume $S = \mathrm{Spec}(\mathbf{Z})$, for the assertion is local on S, and locally on S, the objects under discussion come from corresponding objects over **Z**, which are flat; then the schemes in question are all reduced.

The Plücker morphism embeds $\mathrm{Grass}_{n-a}(Q_X) = \mathrm{Grass}_a(Q_X^*)$ in $\mathbf{P}(\overset{a}{\Lambda} Q_X^*) \cong \mathbf{P}(M)$. In view of (4.3) and (2.8), we obtain an embedding of Z in $\mathbf{P}(M)$ as the closure of the section over $U = X - \sigma_1(A)$ defined by the map $M_U \to O_U$, which is surjective by (2.6); whence, the assertion.

Corollary **(4.5)**. — *The monoidal transformation of* $\mathrm{Grass}_{r-1}(E) = \mathbf{P}(E^*)$ *with the linear space* $\sigma_1(A) = \mathbf{P}(A^*)$ *as center is the projective space bundle* $\mathbf{P}(K^*)$ *over* $Y = \mathbf{P}((E/A)^*)$ *with* $K = \mathrm{Ker}(E_Y \to O_Y(1))$.

Remark **(4.6)**. — By (2.9) (i), the codimension of $f^{-1}(\sigma_{p+1}(A)) = \sigma_{p+1}(A_Y, K)$ in $Z = \mathrm{Grass}_{a-p}(K)$ is $p+1$. It follows that if $p \geq 1$, then f is not the monoidal transformation with center $\sigma_{p+1}(A)$.

Theorem **(4.7)** (splitting bundles). — *Let* k *be an infinite field,* V *a* d*-dimensional quasi-projective* k*-scheme, and* G *an* n*-bundle on* V:

i) *Suppose* $d<n$. *Then* G *contains a subbundle* P *of rank* $n-d$. *In fact,* P *may be taken of the form* $O_X(-p)^{\oplus(n-d)}$ *where* $O_X(1)$ *is ample and* $p \gg 0$.

ii) *Suppose* V *is smooth. Let* $a \geq 1$, *and suppose* $a=1$ *or* $a < n+2-(d/2)$. *Then there exists a monoidal transformation* $f: V' \to V$ *with smooth center* σ *such that* f^*G *contains a subbundle* P *of rank* a. *In fact, given any finite number of irreducible subschemes* V_i *of* V, σ *may be taken such that for all* i, $V_i \cap \sigma$ *has pure codimension* $n-a+1$ *in* V_i, *and is smooth if* V_i *is.*

Indeed, replacing G by $G(p)$ for $p \gg 0$, we may assume V is a twisted subscheme of a suitable $\mathrm{Grass}_n(E)$, and G is the restriction of the universal quotient by virtue of (3.2) (iii). Let A be a general a-dimensional k-subspace of E; it exists by (2.6) and the denseness of the rational points of $\mathbf{A}_k^N$. By (3.5), $\sigma_1(A)$ has appropriate intersection with V and the V_i; take $\sigma = V \cap \sigma_1(A)$. Finally, the assertions result from (4.4), (4.1.2), and the following fact: If $f: Z \to X$ is a monoidal transformation with center Σ, then for any subscheme V of X, the closure of $f^{-1}(V - \Sigma)$ in Z is the monoidal transform of V with center the scheme-theoretic intersection $V \cap \Sigma$.

Remark **(4.8)** (Serre). — Let k be an infinite field, V an arbitrary algebraic k-scheme of dimension d, and G an n-bundle on V. Suppose G is generated by its global sections. If $d<n$, then G contains a trivial bundle A of rank $n-d$; in fact, A may be taken as the bundle generated by $n-d$ general sections.

Indeed, let E be the k-space freely spanned by a finite number of sections generating G, and let $f: V \to \mathrm{Grass}_n(E)$ be the corresponding map. The assertion results from (3.7) with $p=1$ applied to $f(V)$.

5. Smoothing cycles.

Let k be an algebraically closed ground field, and work in the category Σ of connected, smooth quasi-projective k-schemes V with given embeddings in projective space. Let $A(V)=\oplus A^i(V)$ denote the Chow ring of cycles (with integer coefficients) modulo rational equivalence, and for a sheaf G on V, let $c_i(G)\in A^i(V)$ denote its i-th Chern class.

(5.1) Fix $V\in\Sigma$, and let $H\in A^1(V)$ be the class of a hyperplane section. Fix $p\geq 1$, and let $\mathcal{J}_p$ be the set of all (vector) bundles G on V such that

$$c_i(G)=n_iH^i \quad \text{for} \quad i<p$$

where the n_i are suitable integers. For any subset $\mathcal{J}$ of $\mathcal{J}_p$, let $c(\mathcal{J})$ be the set of all $Z\in A^p(V)$ of the form

$$Z=c_p(G)-nH^p$$

for some $G\in\mathcal{J}$ and integer n. Finally, let $\mathcal{J}'_p$ be the set of $G\in\mathcal{J}_p$ such that $\operatorname{rank}(G)\leq\dim(V)$ and $G(-1)$ is generated by its global sections.

Theorem **(5.2)**. — *Under the above conditions, $c(\mathcal{J}'_p)$ is a subgroup of $A^p(V)$ containing $(p-1)!A^p(V)$.*

Indeed, the conclusion results formally from the following three lemmas.

Lemma **(5.3)**. — *$c(\mathcal{J}_p)$ is a group.*

Indeed, given $Z_i=c_p(G_i)-n_iH^p$ $(i=1, 2)$ with $G_i\in\mathcal{J}_p$, let $G=G_1\oplus G_2$. Then $c_i(G)=\sum_j c_j(G_1)c_{i-j}(G_2)$; so, $G\in\mathcal{J}_p$ and $Z_1+Z_2=c(G)-nH^p$ for some n. Now, construct an exact sequence

$$0\to G'_1\to O_V(-m)^{\oplus M}\to G_1\to 0.$$

Then, $c_i(G'_1)=m_iH^i-\sum_{j<i}c_j(G'_1)c_{i-j}c_j(G_1)$ for some m_i; so, by induction on i, $G'_1\in\mathcal{J}_p$ and $-Z_1=c_p(G'_1)-m'H^p$ for some m'.

Lemma **(5.4)**. — *If $Z\in A^p(V)$, then $(p-1)!Z\in c(\mathcal{J}_p)$.*

Indeed, by (5.3), we may assume Z is the class of a closed integral subscheme Z (abusing notation). Using the syzygy theorem, construct a finite, locally free resolution of the form

$$0\to G\to O_V(-m_d)^{\oplus M_d}\to\ldots\to O_V(-m_1)^{\oplus M_1}\to O_Z\to 0.$$

Grothendieck ([2], p. 151, formula (16); [5], p. 53, Lemma (2)) computed that $c_i(O_Z)=0$ $(0<i<p)$ and $c_p(O_Z)=(-1)^p(p-1)!Z$. As in (5.3), it follows that $G\in\mathcal{J}_p$ and $\pm(p-1)!Z=c_p(G)-nH^p$ for some n.

Lemma **(5.5)**. — *$c(\mathcal{J}'_p)=c(\mathcal{J}_p)$.*

Indeed, let $G\in\mathcal{J}_p$. If $\operatorname{rank}(G)>\dim(V)$, then by (4.7) (i), there is an exact sequence

$$0\to O_V(-m)^{\oplus M}\to G\to G'\to 0$$

with G′ a bundle of $\operatorname{rank}=\dim(V)$. As in (5.3), it follows that $G'\in\mathcal{J}_p$ and $c_p(G')=c_p(G)+nH^p$ for some n. Finally, a standard formula shows that for any m, $G(m)\in\mathcal{J}_p$ and $c_p(G(m))=c_p(G)+nH^p$ for some n; whence, the assertion.

Proposition **(5.6)** (Chern). — *Let* $S \in \Sigma$. *Let* E *be an r-bundle on* S, *and* Q *the universal quotient on* $X = \mathrm{Grass}_n(E)$. *Let* A *be a subbundle of* E *of rank* $a \leq n$, *and set* $p = n - a + 1$. *If* A *is trivial, then* $\sigma_1(A)$ *represents* $c_p(Q)$.

Indeed, on $U = X - \sigma_1(A)$, the map $A_U \to Q_U$ is a locally-split injection by virtue of (2.6); let R be its cokernel. Since A is trivial, $c_p(Q_U) = c_p(R)$; however, $\mathrm{rank}(R) = n - a < p$, so $c_p(R) = 0$. In view of (2.8), $\sigma_1(A)$ is irreducible of codimension p. Therefore, $c_p(Q)$ is represented by some multiple m of $\sigma_1(A)$.

To prove $m = 1$, we may restrict S and assume E is free. Choose an $(r - n - 1)$-subbundle C of E such that $A + C$ is an $(a + r - n - 1)$-bundle, and consider the natural inclusion $Y = \mathrm{Grass}_n(E/C) \hookrightarrow X$. In view of (2.13) (i) and the functoriality of Chern classes, we may replace X by Y, and thus assume $r = n + 1$. Then $X = \mathbf{P}(E^*)$, $\sigma_1(A)$ is a linear space by (2.10), and the universal sequence is $0 \to O_X(-1) \to E \to Q \to 0$; whence, the assertion.

Remark **(5.7)**. — Here, (5.6) is viewed in the light of Grothendieck's [2] theory of Chern class; however, its statement was the definitional starting-point of Chow's (unpublished) theory.

Theorem **(5.8)** (smoothing cycles). — *Let k be an algebraically closed field, and* V *a connected, smooth quasi-projective k-scheme of dimension d. Let* Z *be a cycle on* V *of codimension p, and* L *a section of* V *by a linear space of codimension p. Then for a suitable integer n, the cycle* $(p-1)!Z + nL$ *is rationally equivalent to an effective cycle* σ_1 *whose singular locus* σ_2 *is of dimension* $d - (p+1)2$; *in particular,* σ_1 *is smooth if* $\dim(Z) < (d+2)/2$. *Moreover, given any finite number of irreducible subschemes* V_i *of* V, σ_1 *may be taken such that for all i,* $V_i \cap \sigma_1$ (*resp.* $V_i \cap \sigma_2$) *is of pure codimension p* (*resp.* $(p+1)2$) *in* V_i, *and* $V_i \cap (\sigma_1 - \sigma_2)$ *is smooth if* V_i *is.*

Indeed by virtue of (5.2), there exists a twisted embedding of V in a suitable grassmannian over k such that the pull-back of the p-th Chern class of the universal quotient is the rational equivalence class of $(p-1)!Z + nL$ for some n. Therefore the assertions result from (5.6) and (3.3).

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