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CHAPTER 7

The Prime Number Theorem

The proof of the Prime Number Theorem (PNT) by Jacques Hadamard and (independently) Charles de la Vallée Poussin in 1896 is arguably the high water mark of nineteenth century mathematics. Conjectured on the basis of numerical evidence (independently and in somewhat different forms) by Legendre and Gauss at the end of the eighteenth century, PNT asserts that the number $\pi(x)$ of primes less than or equal to x is asymptotic to $x/\log x$ in the sense that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

Since the time of Riemann, it has been understood that the distribution of primes is closely connected with the function theoretic properties of the Riemann zeta function $\zeta(s)$, defined initially for $\operatorname{Re} s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and extended via analytic continuation to $\mathbb{C}$ as a meromorphic function with a single simple pole at $s = 1$. Here the key fact relating the zeta function and PNT is that

$$(*) \qquad \zeta(s) \neq 0 \quad \text{on the line} \quad \operatorname{Re} s = 1.$$

The original proofs of PNT involved integration over infinite contours and therefore required, in addition to the nonvanishing of $\zeta(s)$ on $\operatorname{Re} s = 1$, certain estimates of $\zeta(s)$ near ∞. Subsequent proofs avoided this difficulty but required instead some version of Wiener's Tauberian theory for Fourier integrals (cf., for instance, the proof using the Wiener-Ikehara theorem given in [**C**]). Thus the deduction of PNT from (*) remained highly nontrivial. In 1980, Donald Newman [**N**] discovered an amazingly simple route to deriving PNT from (*). Newman's innovation, in his own words, was "to return to contour integral methods so as to avoid Fourier analysis, and also to use finite contours so as to avoid estimates at infinity." While Newman applied his method to Dirichlet series, we find it more convenient, following Korevaar [**K**], to use it to prove the following Tauberian theorem for Laplace transforms.

THEOREM. *Let f be a bounded measurable function on $[0, \infty)$. Suppose that the Laplace transform*

$$g(z) = \int_0^{\infty} f(t)e^{-zt}\,dt,$$

which is defined and analytic on the open half plane $H = \{z : \operatorname{Re} z > 0\}$, extends analytically to (an open set containing) $\overline{H} = \{z : \operatorname{Re} z \geq 0\}$. Then the improper