integral $\int_0^\infty f(t)dt = \lim_{T\to\infty} \int_0^T f(t)dt$ *converges and coincides with* $g(0)$, *the value of the analytic extension of* g *at* $z = 0$.

REMARK. This result is not new; in fact, it is a special case of a result of Ingham [I], proved by Fourier methods almost half a century earlier. What is of interest here is the simplicity of the proof: by a proper choice of contour and integrand, all previous difficulties are finessed, and one obtains an argument which uses nothing more advanced than the Cauchy integral formula and completely straightforward estimates.

PROOF. Assume that $|f(t)| \le M$ for all $t \ge 0$. For $T > 0$, the function $g_T(z) = \int_0^T f(t)e^{-zt}dt$ is clearly entire. We claim that

$$\lim_{T\to\infty} g_T(0) = g(0). \tag{7.1}$$

To this end, take $R > 0$ large and $\delta = \delta(R) > 0$ so small that g is analytic on the region $D = \{z : |z| \le R,\ \operatorname{Re} z \ge -\delta\}$. Let $\Gamma = \partial D$. Then by Cauchy's Theorem,

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_\Gamma [g(z) - g_T(z)]e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\,dz. \tag{7.2}$$

Let $x = \operatorname{Re} z$. Then for $x > 0$,

$$|g(z) - g_T(z)| = \left|\int_T^\infty f(t)e^{-zt}dt\right| \le M\int_T^\infty e^{-xt}dt = \frac{Me^{-xT}}{x}, \tag{7.3}$$

while

$$\left|e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| = e^{xT}\frac{2|x|}{R^2} \quad \text{for} \quad |z| = R. \tag{7.4}$$

Thus, when $z \in \Gamma_+ = \Gamma \cap \{\operatorname{Re} z > 0\}$, the integrand in (7.2) is bounded in absolute value by $2M/R^2$, and hence

$$\left|\frac{1}{2\pi i}\int_{\Gamma_+} [g(z) - g_T(z)]e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\,dz\right| \le \frac{M}{R}. \tag{7.5}$$

On $\Gamma_- = \Gamma \cap \{\operatorname{Re} z < 0\}$, we consider the integrals involving $g(z)$ and $g_T(z)$ separately. Since g_T is entire, we can replace the contour Γ_- by the semicircle $\Gamma'_- = \{z : |z| = R,\ \operatorname{Re} z < 0\}$. For $x = \operatorname{Re} z < 0$, we have

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}dt\right| \le M\int_{-\infty}^T e^{-xt}dt = \frac{Me^{-xT}}{|x|}; \tag{7.6}$$

so by by (7.4) and (7.6),

$$\left|\frac{1}{2\pi i}\int_{\Gamma'_-} g_T(z)e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\,dz\right| \le \frac{M}{R}. \tag{7.7}$$

Finally, since g is analytic on Γ_-, there exists a constant $K = K(R, \delta)$ such that

$$\left|g(z)\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| \le K \quad \text{on} \quad \Gamma_-.$$

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