Since e^{zT} is bounded on Γ_- and converges uniformly to 0 on compact subsets of $\{\mathrm{Re}\, z < 0\}$ as $T \to \infty$, it follows easily that

$$\lim_{T\to\infty} \left| \frac{1}{2\pi i} \int_{\Gamma_-} g(z) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z}\, dz \right| = 0. \tag{7.8}$$

From (7.2), (7.5), (7.7), and (7.8), we have

$$\overline{\lim_{T\to\infty}}\, |g(0) - g_T(0)| \le \frac{2M}{R}.$$

Since R can be chosen arbitrarily large, this proves (7.1). □

Now let us turn to the actual proof of the Prime Number Theorem, following the concise and elegant development of Zagier [**Z**], which is a model of efficient organization. We begin our discussion with a brief introduction to the Riemann zeta function. Following longstanding tradition, we write the complex variable as $s = \sigma + it$ instead of $z = x + iy$. Define for $\mathrm{Re}\, s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Since $|1/n^s| = 1/n^\sigma$, this series converges absolutely for $\sigma > 1$ and uniformly on $\sigma \ge 1 + \varepsilon$ for each $\varepsilon > 0$. Thus, since the functions $1/n^s = e^{-s \log n}$ are all entire, $\zeta(s)$ is analytic for $\mathrm{Re}\, s > 1$.

LEMMA 7.1. $\zeta(s) - \frac{1}{s-1}$ *extends analytically to* $\mathrm{Re}\, s > 0$.

PROOF. For $\mathrm{Re}\, s > 1$,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^\infty \frac{1}{x^s} dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx.$$

Each summand in the series on the right is evidently an entire function, and the series converges absolutely for $\mathrm{Re}\, s > 0$ since

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| \le \max_{n \le x \le n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| \le \max_{n \le u \le n+1} \left| \frac{s}{u^{s+1}} \right| = \frac{|s|}{n^{\sigma+1}}.$$

Accordingly, convergence is uniform for $\mathrm{Re}\, s \ge \varepsilon$ for each $\varepsilon > 0$, and so the right hand side is analytic for $\mathrm{Re}\, s > 0$. □

REMARK. It is not difficult to show that $\zeta(s) - \frac{1}{s-1}$ actually extends to an entire function. However, we do not require this fact.

Henceforth p denotes a prime number, and sums and products over the index p are taken over all primes. The connection between prime numbers and the zeta function is encoded in the next result, known (for real s) already to Euler.

LEMMA 7.2. $\zeta(s) = \prod_p (1 - 1/p^s)^{-1}$ *for* $\mathrm{Re}\, s > 1$.

PROOF. (Cf. [**A**, p. 213]) Writing p_k for the kth prime, we have

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) \cdots \left(1 - \frac{1}{p_k^s}\right) \zeta(s) = \sum_{2,3,\ldots,p_k \nmid n} \frac{1}{n^s} \to 1$$

as $k \to \infty$. □