It is easy to see that the Euler product for $\zeta(s)$ converges absolutely for $\operatorname{Re} s > 1$ and uniformly for $\operatorname{Re} s \geq 1 + \varepsilon$ for each $\varepsilon > 0$. These facts will be used without further mention below.

Our next result contains the function-theoretic heart of the proof of PNT. Define

$$\Phi(s) = \sum_p \frac{\log p}{p^s}.$$

Since the series converges absolutely for $\operatorname{Re} s > 1$ and uniformly for $\operatorname{Re} s \geq 1 + \varepsilon$ for each $\varepsilon > 0$, Φ is analytic in $\operatorname{Re} s > 1$.

LEMMA 7.3. *$\Phi(s) - \frac{1}{s-1}$ extends analytically to $\operatorname{Re} s \geq 1$, and $\zeta(s) \neq 0$ for $\operatorname{Re} s = 1$.*

PROOF. The proof of Lemma 7.2 shows that $\zeta(s) \neq 0$ for $\operatorname{Re} s > 1$. A simple calculation based on the product representation then yields

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s - 1)}. \tag{7.9}$$

The last term on the right converges and defines an analytic function for $\operatorname{Re} s > 1/2$, so it follows from Lemma 7.1 that $\Phi(s)$ extends to a meromorphic function on $\operatorname{Re} s > 1/2$ with poles only at $s = 1$ and at the zeros of $\zeta(s)$ and that $\Phi(s) - \frac{1}{s-1}$ is analytic at $s = 1$. Thus, it remains only to show that $\zeta(s)$ does not vanish for $\operatorname{Re} s = 1$.

To this end, recall that if a meromorphic function f vanishes to (exact) order k at s_0, then

$$\lim_{s \to s_0} (s - s_0) \frac{f'(s)}{f(s)} = \operatorname{Res}\left(\frac{f'}{f}, s_0\right) = k \tag{7.10}$$

and, similarly, that if f has a pole of order k at s_0,

$$\lim_{s \to s_0} (s - s_0) \frac{f'(s)}{f(s)} = \operatorname{Res}\left(\frac{f'}{f}, s_0\right) = -k. \tag{7.11}$$

Suppose now that $\zeta(s)$ has a zero of order $\mu \geq 0$ at $s = 1 + i\alpha$ ($\alpha \neq 0$, $\alpha \in \mathbb{R}$); since $\zeta(s)$ is real for real s, it follows that $\zeta(s)$ has a zero of the same multiplicity at $1 - i\alpha$. Denoting the multiplicity of the zeros (if any) at $s = 1 \pm 2i\alpha$ by $\nu \geq 0$ and applying (7.10) and (7.11) to the function $\Phi(s)$, which differs from $-\zeta'(s)/\zeta(s)$ by a function analytic on $\operatorname{Re} s > 1/2$, we obtain

$$\begin{gathered} \lim_{\varepsilon \to 0^+} \varepsilon \Phi(1 + \varepsilon) = 1 \qquad \text{and} \\ \lim_{\varepsilon \to 0^+} \varepsilon \Phi(1 + \varepsilon \pm i\alpha) = -\mu \qquad \lim_{\varepsilon \to 0^+} \varepsilon \Phi(1 + \varepsilon \pm 2i\alpha) = -\nu. \end{gathered} \tag{7.12}$$

But for $\varepsilon > 0$,

$$\sum_{k=-2}^{2} \binom{4}{2+k} \Phi(1 + \varepsilon + ik\alpha) = \sum_p \frac{\log p}{p^{1+\varepsilon}} (p^{i\alpha/2} + p^{-i\alpha/2})^4 \geq 0, \tag{7.13}$$

since the quantity in parentheses on the right is real. Multiplying (7.13) by ε and using (7.12) to calculate the limit of the left hand side as $\varepsilon \to 0+$, we obtain $-2\nu - 8\mu + 6 \geq 0$. Thus $\mu = 0$, i.e., $\zeta(1 + i\alpha) \neq 0$. This concludes the proof of Lemma 7.3. $\square$