We have completed the preparations for proving PNT. The rest of the proof focuses on the function

$$\theta(x) = \sum_{p \le x} \log p.$$

We shall show that $\theta(x) \sim x$, i.e., $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$. This easily implies PNT since

$$\theta(x) = \sum_{p \le x} \log p \le \sum_{p \le x} \log x = \pi(x) \log x,$$

while for any $\varepsilon > 0$,

$$\theta(x) \ge \sum_{x^{1-\varepsilon} \le p \le x} \log p \ge \sum_{x^{1-\varepsilon} \le p \le x} (1-\varepsilon) \log x = (1-\varepsilon) \log x [\pi(x) + O(x^{1-\varepsilon})].$$

First, following Chebyshev, we prove

LEMMA 7.4. $\theta(x) = O(x)$.

PROOF. For n a positive integer, we have

$$2^{2n} = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \ge \binom{2n}{n} \ge \prod_{n<p\le 2n} p = e^{\theta(2n)-\theta(n)},$$

so that $\theta(2n) - \theta(n) \le 2n \log 2$. It follows that

$$\begin{aligned}\theta(x) - \theta(x/2) = \theta(x) - \theta([x/2]) &\le \log x + \theta(2[x/2]) - \theta([x/2]) \\ &\le \log x + 2[x/2] \log 2 \le (1 + \log 2)x.\end{aligned}$$

Summing successively over $x, x/2, \dots, x/2^r$, where $2^r > x$, we obtain $\theta(x) \le 2(1 + \log 2)x$. □

LEMMA 7.5. *The integral* $\int_1^\infty [\theta(x) - x]/x^2 \, dx$ *converges.*

PROOF. This follows directly from the Tauberian theorem of Section 1 applied to the function $f(t) = \theta(e^t)e^{-t} - 1$, which by Lemma 7.4 is bounded. Indeed, using Lemma 7.4 again, we have for $\operatorname{Re} s > 1$,

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = \int_1^\infty \frac{d\theta(x)}{x^s} = s \int_1^\infty \frac{\theta(x)}{x^{s+1}} dx = s \int_0^\infty e^{-st} \theta(e^t) dt,$$

so that

$$\begin{aligned}g(s) = \int_0^\infty f(t) e^{-st} dt = \int_0^\infty [\theta(e^t)e^{-t} - 1] e^{-st} dt &= \frac{\Phi(s+1)}{s+1} - \frac{1}{s} \\ &= \frac{1}{s+1}\left[\Phi(s+1) - \frac{1}{s} - 1\right],\end{aligned}$$

which extends analytically to $\operatorname{Re} s \ge 0$ by Lemma 7.3. Thus

$$\int_1^\infty \frac{\theta(x) - x}{x^2} dx = \int_0^\infty [\theta(e^t)e^{-t} - 1] dt = \int_0^\infty f(t) dt,$$

which converges. □