To complete the proof of PNT, let us show how Lemma 7.5 implies that $\theta(x) \sim x$. Assume that for some $\lambda > 1$, there exist arbitrarily large x with $\theta(x) \geq \lambda x$. Then, since θ is nondecreasing, for each such x,

$$\int_x^{\lambda x} \frac{\theta(t) - t}{t^2}\, dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt = \int_1^{\lambda} \frac{\lambda - t}{t^2} dt > 0,$$

which implies the divergence of $\int_1^\infty [\theta(t) - t]/t^2 dt$, contrary to Lemma 7.5. Similarly, if $\theta(x) \leq \lambda x$ for some $\lambda < 1$ and arbitrarily large x, we would have

$$\int_{\lambda x}^{x} \frac{\theta(t) - t}{t^2}\, dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2} dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2} dt < 0,$$

which would again contradict the convergence of $\int_1^\infty [\theta(t) - t]/t^2 dt$. Thus

$$\lim_{x \to \infty} \theta(x)/x = 1,$$

and the proof is done.

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