MAT4250 EXERCISE SHEET 4

Exercise 1. Let p be an odd prime. Show the *supplementary quadratic reciprocity law*

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

Exercise 2. Is 43 a quadratic residue modulo 3319?

Exercise 3. Let $p \equiv 1 \pmod 4$ be a rational prime, and consider the fields $L = \mathbf{Q}(\zeta_p)$ and $K = \mathbf{Q}(\sqrt{p})$.

(a) Show that $K \subseteq L$.

Now let q be a rational prime satisfying $q^n \equiv -1 \pmod p$ for some positive odd integer n.

(b) Determine the factorization type of $q\mathcal{O}_K$ and $q\mathcal{O}_L$.

Exercise 4. Let L/K be a separable field extension. Show that $\Omega^1_{L/K} = 0$.

Exercise 5. Give an example of an extension of number fields L/K such that $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ is not a projective $\mathcal{O}_L$-module.

Exercise 6. Let A be a commutative ring.

(a) Let M be a free A-module of rank n, with basis $e_1, \ldots, e_n$. Then $\bigwedge^k M$ is the free A-module of rank $\binom{n}{k}$, with basis $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq n$. (Thus, in particular, $\bigwedge^n A^n \cong A$, with basis $e_1 \wedge \cdots \wedge e_n$.)

(c) Let M be a free A-module of rank n, and let $\phi\colon M \to M$ be an A-linear endomorphism. Then the induced map $\bigwedge^n \phi\colon \bigwedge^n M \to \bigwedge^n M$ is multiplication by $\det \phi$.

Exercise 7. Let A be a Dedekind ring. Use the characterization of finitely generated projective modules over A to show that $K_0(A) \cong \mathbf{Z} \oplus \mathrm{Cl}_A$. If A is the ring of integers in a number field, show that $\widehat{K}_0(A) \cong \mathbf{Z} \oplus \widehat{\mathrm{Cl}}_A$.