1. We must show that $\mathbb{Q}(\sqrt{d})$ is contained in a cyclotomic extension of $\mathbb{Q}$. (1)

Assume first that $d = p$ is a prime number.

If $p = 2$, then $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\zeta_8)$ (indeed,

$\zeta_8 + \overline{\zeta_8} = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2} \in \mathbb{Q}(\zeta_8)$),

so we may assume that p is odd.

By the proof of Proposition 10.5,

we then have $\mathbb{Q}(\sqrt{p^*}) \subseteq \mathbb{Q}(\zeta_p)$, where

$p^* = (-1)^{\frac{p-1}{2}} p$ (in fact, this is the unique quadratic field contained in $\mathbb{Q}(\zeta_p)$).

So
$$\mathbb{Q}(\sqrt{p}) \subseteq \begin{cases} \mathbb{Q}(\zeta_p), & p \equiv 1 \ (4) \\ \mathbb{Q}(\zeta_p, i), & p \equiv 3 \ (4) \end{cases}$$

In general, for $d = \pm 2^{\delta} p_1 \cdots p_n$, we then have

$$\mathbb{Q}(\sqrt{d}) \subseteq \mathbb{Q}(\zeta_8^{\delta}, \zeta_{p_1}, \ldots, \zeta_{p_n}, i),$$

where $\delta = 0$ or 1 and the p_i's are odd.

②

2. a) Let $G = \mathbb{F}_q^*$, and let m be the least common multiple of the orders of the elements of G. Then G is cyclic $\Leftrightarrow$ $m = |G|$, and $x^m = 1$ for all $x \in G$.

This equation has at most m solutions, which shows $m \geq |G|$. Since of course $m \leq |G|$ we thus have $m = |G|$, so G is cyclic.

b) Let $L/\mathbb{F}_q$ be a degree n - extension of $\mathbb{F}_q$. Then $|L| = q^n$. Consider the Frobenius homomorphism

$$\mathrm{Frob}_q : L \longrightarrow L,\quad x \longmapsto x^q.$$

Then $\mathrm{Frob}_q(x) = x \;\forall\, x \in \mathbb{F}_q$, so Frob_q fixes $\mathbb{F}_q$. Moreover, if $\mathrm{Frob}_q(x) = \mathrm{Frob}_q(y)$, then $(x-y)^q = 0$, hence $x = y$. So Frob_q is injective, hence bijective since L is a finite dimensional $\mathbb{F}_q$ - vector space.

Now,
$$L^{\mathrm{Frob}_q} = \{x \in L \mid x^q = x\} = \{\text{roots of } X^q - X\} = \mathbb{F}_q,$$

so the fixed field of Frob_q is precisely $\mathbb{F}_q$.

This means that $L/\mathbb{F}_q$ is Galois, with Galois group generated by Frob_q. Since Frob_q has order n, it follows that $\mathrm{Gal}(L/\mathbb{F}_q) \cong \mathbb{Z}/n$.

This moreover shows that there is a unique degree n extension of $\mathbb{F}_q$ for each $n \geq 1$, which is given as the splitting field of $X^{q^n} - X$.

3\. a) By Dirichlet's unit theorem,

Γ_K is a full lattice in $\hat{H}(K)^0 \cong \mathbb{R}^{r+s-1}$.

So $\hat{H}(K)^0/\Gamma_K \cong \mathbb{R}^{r+s-1}/\mathbb{Z}^{r+s-1} = \mathbb{T}^{r+s-1}$, the

$(r+s-1)$-dimensional real torus, which is a compact Lie group.

b) Recall that we have an exact sequence

$$0 \longrightarrow \hat{H}(K)/\Gamma_K \xrightarrow{a} \widehat{CH}^1(\mathcal{O}_K) \xrightarrow{\zeta} Cl_K \longrightarrow 0.$$

Since $a([g]) = [0, -g]$, the map a induces a map $a: \hat{H}(K)^0/\Gamma_K = T^0 \longrightarrow \widehat{CH}^1(\mathcal{O}_K)^0$ which is still injective.

Moreover, since K has at least one embedding into $\mathbb{C}$ (or $\mathbb{R}$), it follows that $\zeta|_{\widehat{CH}^1(\mathcal{O}_K)^0}$ is still surjective. (indeed, for $[\mathfrak{a}] = [\prod \mathfrak{p}^{a_\mathfrak{p}}] \in Cl_K$, we find an element $[\sum (-a_\mathfrak{p})\mathfrak{p}, g] \in \widehat{CH}^1(\mathcal{O}_K)^0$ mapping to $[\mathfrak{a}]$ by defining $g = (g_\sigma)$ so that $\sum g_\sigma = -\sum (-a_\mathfrak{p}) \log N(\mathfrak{p})$).

Clearly $\operatorname{im} a = \ker \zeta$, so the sequence is exact.

In fact, since $T^0 \cong \mathbb{R}^{r+s-1}/\mathbb{Z}^{r+s-1}$ is divisible, it follows that the sequence is split exact, so $\widehat{CH}^1(\mathcal{O}_K)^0 \cong Cl_K \times T^0$

(4) c)

- $K = \mathbb{Q}$: Then $r+s-1 = 1+0-1 = 0$ & $Cl_K = 0$,

  so $\widehat{CH}^1(\mathcal{O}_K)^0 = 0$ by the exact sequence

- $K = \mathbb{Q}(\sqrt{2})$: $r+s-1 = 2+0-1 = 1$,

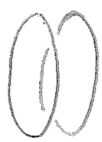
  $Cl_{\mathbb{Q}(\sqrt{2})} = 0$,

  so $\widehat{CH}^1(\mathcal{O}_K)^0 \cong T^0 = \mathbb{R}/\mathbb{Z} = S^1$, a circle.

- $K = \mathbb{Q}(\theta)$: $r+s-1 = 1+1-1 = 1$ & $Cl_K = \mathbb{Z}/2$ by Exercise sheet 2.

  So $\widehat{CH}^1(\mathcal{O}_K)^0 \cong \mathbb{Z}/2 \times \mathbb{R}/\mathbb{Z}$.

Thus we get:

K	$\widehat{CH}^1(\mathcal{O}_K)^0$	
$\mathbb{Q}$	$\cdot$	(0)
$\mathbb{Q}(\sqrt{2})$		$(\mathbb{R}/\mathbb{Z})$
$\mathbb{Q}(\theta)$		$(\mathbb{Z}/2 \times \mathbb{R}/\mathbb{Z})$

4. Recall that $i_* : \widehat{CH}^1(\mathcal{O}_L) \to \widehat{CH}^1(\mathcal{O}_K)$ is given by

$$i_*\left(\sum a_{\mathfrak{q}} \mathfrak{q} , (g_\tau)_{\tau \in \Sigma_L} \right) = \left(\sum a_{\mathfrak{q}} N_{L/K}(\mathfrak{q}), \left(\sum_{\tau|_K = \sigma} g_\tau \right)_{\sigma \in \Sigma_K} \right).$$

We must show that $i_* \widehat{Rat}^1(\mathcal{O}_L) \subseteq \widehat{Rat}(\mathcal{O}_K)$.

Let $f \in L^*$, then

$$i_* \left(\widehat{div}_{\mathcal{O}_L}(f) \right) = \left(\sum_{\mathfrak{q}} \mathrm{ord}_{\mathfrak{q}}(f) \, N_{L/K}(\mathfrak{q}), \left(\sum_{\tau|_K = \sigma} -\log|\tau(f)| \right)_\sigma \right)$$

$$= \left(div\,(N_{L/K}(f)), \left(-\log \left| \prod_{\tau|_K = \sigma} \tau(f) \right| \right)_{\sigma \in \Sigma_K} \right)$$

$$= \left(div(N_{L/K}(f)), \left(-\log \left| \sigma\left(\prod_{\tau \in \Sigma_L} \tau(f) \right) \right| \right)_{\sigma \in \Sigma_K} \right)$$

$$= \left(div(N_{L/K}(f)), (-\log|\sigma(N_{L/K}(f))|)_{\sigma \in \Sigma_K} \right)$$

$$= \widehat{div}\,(N_{L/K}(f)).$$