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1. See the proof of Theorem 8.6.

in Neukirch.

2. We have $43 \equiv 3 \ (4)$ and $3319 \equiv 3 \ (4)$.

Hence
$$\left(\frac{43}{3319}\right) = -\left(\frac{3319}{43}\right)$$

by quadratic reciprocity.

Moreover, $3319 \equiv 8 \bmod 43$, so

$$-\left(\frac{3319}{43}\right) = -\left(\frac{8}{43}\right) = -\left(\frac{2}{43}\right)^3$$

$$= -\left(\frac{2}{43}\right)$$

$$= -(-1)^{\frac{43^2-1}{8}}$$

$$= 1.$$

So $\left(\frac{43}{3319}\right) = 1$, i.e., 43 is

a square mod. 3319.

3 a) By the proof of Theorem 8.6, we know that

$p^* = (-1)^{\frac{p-1}{2}} p \overset{p \equiv 1\,(4)}{=} p$ is a square in $\mathbb{Q}(\zeta_p)$,

hence $K \subseteq L$.

b) We have that $q\mathcal{O}_K$ splits $\Leftrightarrow$ p is a square mod q.

Now,

$$1 = (-1)^{\frac{p-1}{2}} = \left(\frac{-1}{p}\right) = \left(\frac{q^n}{p}\right) \overset{\text{Multiplicativity of the Legendre symbol}}{=} \left(\frac{q}{p}\right)^n \underset{\text{By quadratic reciprocity, Since } p \equiv 1\,(4).}{=} \left(\frac{p}{q}\right)^n$$

Since n is odd, it follows that $\left(\frac{p}{q}\right) = 1$, i.e., p is a square mod q.

Hence q is totally split in K.

By Prop. 10.5 in Neukirch, q splits into an even number of prime ideals in L.

4. See the example in the lecture notes on G-R-R.

5. Let $K = \mathbb{Q}$ and $L = \mathbb{Q}(i)$.

Then, in $\Omega^1_{\mathbb{Z}[i]/\mathbb{Z}}$ we have

$$0 = d(-1) = d(i^2) = 2i\,di,$$

so di is killed by $2i \in \mathbb{Z}[i]$, i.e., $\Omega^1_{\mathbb{Z}[i]/\mathbb{Z}}$ is torsion. So $\Omega^1_{\mathbb{Z}[i]/\mathbb{Z}}$ is not a flat $\mathbb{Z}[i]$-module, hence not projective either.

6. a) Define the map $M^{\times k} \xrightarrow{\varphi} M^{\otimes k}$ by

$$(m_1, \dots, m_k) \longmapsto \sum_{\sigma \in \Sigma_k} \operatorname{sgn}(\sigma)\, m_{\sigma(1)} \otimes \dots \otimes m_{\sigma(k)}$$

Then φ is alternating, hence induces a map

$$\varphi_k : \Lambda^k M \longrightarrow M^{\otimes k}, \quad m_1 \wedge \dots \wedge m_k \longmapsto \sum_{\sigma} \operatorname{sgn}(\sigma)\, m_{\sigma(1)} \otimes \dots \otimes m_{\sigma(k)}.$$

We aim to show that φ_k is injective.

This will conclude the proof: Indeed, the elements $e_{i_1} \wedge \dots \wedge e_{i_k}$ for $1 \le i_1 < \dots < i_k \le n$ span $\Lambda^k M$; we need to show that they are linearly independent. This follows from injectivity of φ_k, since $M^{\otimes k}$ is free on basis $e_{i_1} \otimes \dots \otimes e_{i_k}$ for $1 \le i_1, \dots, i_k \le n$.

We may assume $k \geq 2$.

④ Suppose that $\omega \in \Lambda^k M$ is such that

$$\varphi_k(\omega) = 0.$$

Write

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Then

$$\varphi_k(\omega) = \sum a_{i_1 \cdots i_k} \left(\sum_\sigma \operatorname{sgn}(\sigma) e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}} \right) = 0. \qquad (*)$$

Now the set $\{ e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}} \}$ for varying σ & i_j's form a <u>basis</u> for $M^{\otimes k}$; in particular they are linearly independent. Hence the relation $(*)$ gives $a_{i_1 \cdots i_k} = 0 \ \forall i_j$.

b) Let $e_1, \ldots, e_n$ be a basis for M, so that $\Lambda^n M$ has basis $e_1 \wedge \cdots \wedge e_n$. The induced map

$$\Lambda^n \varphi : \Lambda^n M \longrightarrow \Lambda^n M$$

is given by $(\Lambda^n \varphi)(m_1 \wedge \cdots \wedge m_n) = \varphi(m_1) \wedge \cdots \wedge \varphi(m_n)$;

we must show that $(\Lambda^n \varphi)(e_1 \wedge \cdots \wedge e_n) = (\det \varphi)(e_1 \wedge \cdots \wedge e_n)$.

Let $(a_{ij})_{ij}$ be the matrix of φ, so that

$$(\Lambda^n \varphi)(e_1 \wedge \cdots \wedge e_n) = \left(\sum_{i=1}^n a_{i1} e_i \right) \wedge \cdots \wedge \left(\sum_{i=1}^n a_{in} e_i \right). \qquad (**)$$

Multiplying out (* *), we may ignore all terms with equal indices, as the wedge product vanishes in that case.
It follows that we are left with:

$$(\Lambda^n \varphi)(e_1 \wedge \cdots \wedge e_n) = \sum_{\sigma \in \Sigma_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} \left(e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} \right).$$

By skew-symmetry

$$= \sum_{\sigma \in \Sigma_n} (\operatorname{Sgn} \sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n} (e_1 \wedge \cdots \wedge e_n)$$

$$= (\det \varphi)(e_1 \wedge \cdots \wedge e_n).$$

7. We know that a projective A-module P of rank n is isomorphic to $A^{n-1} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an ideal depending only on its class in Cl_A.

Hence the map

$$K_0(A) \ni [A^{n-1} \oplus \mathfrak{a}] \longrightarrow (n, [\mathfrak{a}]) \in \mathbb{Z} \oplus Cl_A$$

defines an isomorphism $K_0(A) \cong \mathbb{Z} \oplus Cl_A$.

The statement about $\hat{K}_0(\mathcal{O}_K)$ was proved in class, using e.g. the arithmetic Chern character.