MAT4250 EXERCISE SHEET 12

1. Hilbert class fields, conductors

Exercise 1. Let $K \subseteq K' \subseteq K''$ be a tower of Hilbert class fields, i.e., K' is the Hilbert class field of K, and K'' is the Hilbert class field of K'. Show that $\mathrm{Gal}(K''/K')$ is the commutator subgroup of $\mathrm{Gal}(K''/K)$.

Exercise 2.

(a) Let $K/\mathbb{Q}$ be a finite abelian extension and let K' be the Hilbert class field of K. If K' is also abelian over $\mathbb{Q}$, show that we have the equality of conductors

$$\mathfrak{f}(K'/\mathbb{Q}) = \mathfrak{f}(K/\mathbb{Q}).$$

(b) Compute the conductor of $\mathbb{Q}(\sqrt{-1}, \sqrt{-5})/\mathbb{Q}$.

Exercise 3. Let p be an odd prime and let K be the Hilbert class field of $\mathbb{Q}(\zeta_p)$. If $K/\mathbb{Q}$ is abelian, what can you say about the class group of $\mathbb{Q}(\zeta_p)$?

2. Dirichlet L-series

Exercise 4. Let χ be the unique nontrivial Dirichlet character modulo 4, i.e.,

$$\chi(n) = \begin{cases} 1, & n \equiv 1 \pmod 4 \\ -1, & n \equiv 3 \pmod 4. \end{cases}$$

Show that $L(1, \chi) = \frac{\pi}{4}$.

3. Hilbert symbols (continued from Set 9)

In Set 9 we defined the second K-group $K_2(F)$ of a field F, and, for $F = \mathbb{Q}$, a map

$$\partial = \bigoplus_{v \leq \infty} \partial_v \colon K_2(\mathbb{Q}) \to \mu_2 \oplus \bigoplus_{p \geq 2} \mathbb{F}_p^\times.$$

Here $\partial_\infty \colon K_2(\mathbb{Q}) \to \mu_2$ denotes the Hilbert symbol at ∞, while $\partial_p \colon K_2(\mathbb{Q}) \to \mathbb{F}_p^\times$ is the tame symbol at p. We aim to show that ∂ is an isomorphism.

Exercise 5. Define the following subgroups of $K_2(\mathbb{Q})$:

$$\Lambda_\infty = \langle \{-1, -1\} \rangle$$
$$\Lambda_0 = \langle \{a, b\} : a, b \in \mathbb{Z}_{>0} \rangle$$
$$\Lambda_p = \langle \{a, b\} : 1 \leq a, b \leq p \rangle$$

(a) Show that $K_2(\mathbb{Q}) = \Lambda_0 \oplus \Lambda_\infty$, and $\Lambda_0 = \bigcup_p \Lambda_p$.

Note that ∂_∞ gives an isomorphism $\Lambda_\infty \cong \mu_2$, while $\partial_\ell(\Lambda_0) \subseteq \bigoplus_p \mathbb{F}_p^\times$ for all ℓ. We will show by induction that ∂ induces an isomorphism $\Lambda_p \cong \bigoplus_{\ell \leq p} \mathbb{F}_p^\times$.

(b) Show directly that $\Lambda_2 = \{1\}$.

For the induction step, it suffices to show that $\Lambda_p/\Lambda_q \cong \mathbb{F}_p^\times$, where q is the greatest prime smaller than p. For this, define maps

$$\phi\colon \Lambda_p/\Lambda_q \rightleftarrows \mathbb{F}_p^\times : \psi$$

by $\phi(\{a,b\}) = \partial_p(\{a,b\})$ and $\psi([x]) = \{x,p\} \pmod{\Lambda_q}$, where x is the representative for $[x]$ such that $1 \leq x < p$.

(c) Show that ϕ and ψ are well defined inverse isomorphisms.

The above structure theorem for $K_2(\mathbb{Q})$ was first proven by Tate, and it gives in fact yet another proof of quadratic reciprocity (using no class field theory). Indeed, there is one symbol on $\mathbb{Q}$ we have not yet made use of, namely the Hilbert symbol from $\mathbb{Q}_2$. It gives a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}^\times \times \mathbb{Q}^\times & \longrightarrow & K_2(\mathbb{Q}) \cong \mu_2 \oplus \bigoplus_p \mathbb{F}_p^\times \\ {\scriptstyle (-,-)_2} \big\downarrow & \swarrow {\scriptstyle \bigoplus_{2<v\leq\infty} \psi_v} & \\ \mu_2, & & \end{array}$$

i.e., $(a,b)_2 = \prod_{2<v\leq\infty} \psi_v(a,b)$. Now there are not many homomorphisms from a finite cyclic group to μ_2; in fact the ψ_v's have to be either the trivial map or the Hilbert symbols at the various places. Thus we obtain

$$(a,b)_2 = \prod_{2<v\leq\infty} (a,b)_v^{\delta_v}$$

where δ_v is 0 or 1. By plugging in various primes you can check[1] that in fact all $\delta_v = 1$, hence we obtain the product formula

$$\prod_{2\leq v\leq\infty} (a,b)_v = 1$$

for Hilbert symbols on $\mathbb{Q}$, which is an equivalent formulation of quadratic reciprocity.

[1]The cases when $p \not\equiv 1 \pmod 8$ follow directly from properties of the formulas for the Hilbert symbols; the case $p \equiv 1 \pmod 8$ is nontrivial since the left hand side will be 1 no matter what else you plug in. In this case one argues by contradiction using a lemma from Gauss' first proof of quadratic reciprocity, namely that there exists a prime $q < p$ such that $\left(\frac{p}{q}\right) = -1$.