MAT4250 EXERCISE SHEET 3

1. Absolute values

Exercise 1. Show that an absolute value on K is nonarchimedean (i.e., satisfies the strong triangle inequality) if and only if (the image of) $\mathbb{Z}$ is bounded in K.

Thus, in particular, a field of characteristic p can only have nonarchimedean absolute values.

Exercise 2. If $(K, |\cdot|)$ is a field equipped with an absolute value, then K becomes a metric space by setting $d(x, y) = |x - y|$. If $|\cdot|$ satisfies the strong triangle inequality, then we call the resulting metric space an *ultrametric space*. We study some of the properties of ultrametric spaces. Thus, suppose that $|\cdot|$ satisfies the strong triangle inequality.

(a) If $|x| \neq |y|$, show that $|x + y| = \max\{|x|, |y|\}$.
(b) Show that every triangle in K is isosceles.
(c) Show that every point in a ball $B(x, \epsilon)$ in K is a center of the same ball.
(d) Prove the *nonarchimedean convergence criterion*: a series $\sum_{n=1}^{\infty} a_n$ from K converges if and only if $\lim_{n\to\infty} |a_n| = 0$.

Exercise 3.

(a) Show that any nonarchimedean valuation $v\colon K^\times \to G$ on a number field K is discrete.
(b) Give an example of a field K possessing a nondiscrete valuation.

2. Local fields

Exercise 4.

(a) Show that $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, and that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.
(b) Show that $\mathbb{Z}_p$ is totally disconnected.

Exercise 5. Show that the polynomial $(X^2 - 2)(X^2 - 17)(X^2 - 34)$ has a root in all completions of $\mathbb{Q}$, but not in $\mathbb{Q}$.

Exercise 6 (Structure of $\mathbb{Q}_p^\times$)**.** Let $\mu(\mathbb{Q}_p)$ denote the roots of unity in $\mathbb{Q}_p$, and, for $n \geq 1$, let U_p^n denote the multiplicative group $U_p^n = 1 + p^n\mathbb{Z}_p$.

(a) Define homomorphisms

$$\log\colon (U_p^1, \cdot) \rightleftarrows (\mathbb{Z}_p, +) : \exp$$

and show that these are inverse isomorphisms for $p \neq 2$.

For $p = 2$, show that $U_2^1 \cong \mu_2 \times U_2^2$.

(b) Show that $\mu(\mathbb{Q}_p) = \mu_{p-1}$ for p odd, while $\mu(\mathbb{Q}_2) = \mu_2$.
(c) Let p be any prime. Prove that $\mathbb{Z}_p^\times \cong \mu_{p-1} \times U_p^1$, and that $\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mu_{p-1} \times U_p^1$.
(d) Let $p \neq 2$. Show that $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, with a system of representatives given by $\{1, u, p, up\}$, where $u \in \mathbb{Z}_p^\times$ is a nonsquare modulo p.
(e) Show that $\mathbb{Q}_2^\times/\mathbb{Q}_2^{\times 2} \cong (\mathbb{Z}/2)^3$, with a system of representatives given by $\{\pm 1, \pm 2, \pm 5, \pm 10\}$.