MAT4250 EXERCISE SHEET 4

1. Norms of ideals

Exercise 1. Let $A \subseteq B$ be an extension of Dedekind rings, with fraction fields $K \subseteq L$. For $\mathfrak{a}$ a fractional ideal of B, let $\operatorname{Nm}(\mathfrak{a})$ be the fractional ideal of A generated by $N_{L/K}(x)$ for $x \in B$.

(a) Show that Nm commutes with localization, i.e., $\operatorname{Nm}(S^{-1}\mathfrak{a}) = S^{-1}\operatorname{Nm}(\mathfrak{a})$.
(b) Show that $\operatorname{Nm}(\mathfrak{a}\mathfrak{b}) = \operatorname{Nm}(\mathfrak{a})\operatorname{Nm}(\mathfrak{b})$.
(c) $\operatorname{Nm}(\mathfrak{q}) = \mathfrak{p}^f$, where $\mathfrak{q}$ is a prime of B, $\mathfrak{p} = \mathfrak{q} \cap A$ and $f = f(\mathfrak{q}/\mathfrak{p}) = [k(\mathfrak{q}) : k(\mathfrak{p})]$.

Exercise 2. Let $K = \mathbb{Q}(\sqrt{-5})$. Compute $\operatorname{Nm}(\mathfrak{p})$ for $\mathfrak{p} = (2, 1 + \sqrt{-5})$ and $(3, 1 + \sqrt{-5})$.

2. Class fields

Exercise 3. Show using Furtwängler's theorem that there are no finite unramified abelian extensions of $\mathbb{Q}(\sqrt{-1})$.

Exercise 4. Show that the Hilbert class field of $\mathbb{Q}(\sqrt{-5})$ is $\mathbb{Q}(\sqrt{-5}, \sqrt{-1})$.

Exercise 5. Let $K = \mathbb{Q}$. Compute the ray class group $C_{\mathfrak{m}}$ when $\mathfrak{m}$ is (5) and $(5)\infty$.

Exercise 6. Let K be an imaginary quadratic number field ramified at t finite primes. Show that $|\operatorname{Cl}(K)/2\operatorname{Cl}(K)| = 2^{t-1}$.