MAT4250 EXERCISE SHEET 5

1. Global class field theory

Exercise 1.

(a) Let $K = \mathbb{Q}$ and let m be a natural number. Show that $C_{\mathfrak{m}} \cong (\mathbb{Z}/m)^\times/\{\pm 1\}$ if $\mathfrak{m} = (m)$, and $C_{\mathfrak{m}} \cong (\mathbb{Z}/m)^\times$ if $\mathfrak{m} = (m)\infty$.

(b) Show that the ray class field for $\mathfrak{m} = (m)$ is $\mathbb{Q}(\zeta_m)^+ := \mathbb{Q}(\zeta_m + \overline{\zeta}_m)$, and that the ray class field for $\mathfrak{m} = (m)\infty$ is $\mathbb{Q}(\zeta_m)$.

(c) Use Artin reciprocity for $\mathbb{Q}$ to prove the Kronecker–Weber Theorem: every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension.

2. Inverse limits

Exercise 2. Show that inverse limits are unique up to unique isomorphism.

Exercise 3. Prove that in the category of sets, or topological spaces, or groups, the inverse limit is given by

$$\varprojlim S_\alpha = \left\{ (x_\alpha)_\alpha \in \prod_\alpha S_\alpha : \phi_{\alpha\beta}(x_\beta) = x_\alpha \right\}.$$

Exercise 4. Prove that $\mathbb{Z}_p \cong \varprojlim \mathbb{Z}/p^n$ as topological groups.

Exercise 5.

(a) Show that inverse limits of sets commute with products.

(b) Recall that $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n$. Show that $\widehat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$.

(c) Is $\widehat{\mathbb{Z}}$ an integral domain?

Exercise 6. $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$.

3. Higher ramification groups

Let L_w/K_v be a finite Galois extension of local fields, and let $G = \mathrm{Gal}(L_w/K_v)$. Write $\mathfrak{p}_w = (\pi_w)$ for the maximal ideal of $\mathcal{O}_{L_w}$.

We define a filtration on G,

$$\cdots \subseteq G_i \subseteq \cdots \subseteq G_1 \subseteq G_0 \subseteq G_{-1} = G$$

as follows. Let

$$G_i = \{\sigma \in G : \mathrm{ord}_{\pi_w}(\sigma(\alpha) - \alpha) \geq i + 1 \text{ for all } \alpha \in L_w^\times\}.$$

Exercise 7.

(a) Show that G_0 is the inertia group of L_w/K_v.

(b) $G_i = 1$ for $i \gg 0$.

(c) $G/G_0 \cong \mathrm{Gal}(k(w)/k(v))$, where $k(w)/k(v)$ is the residue field extension.

(d) L_w/K_v is unramified $\iff G_0 = 1$, and is tamely ramified $\iff G_1 = 1$.