MAT4250 EXERCISE SHEET 6

1. Local class field theory

Exercise 1. Find all subgroups of $\mathbb{Q}_2^\times$ that are norm groups from a quadratic extension of $\mathbb{Q}_2$. To which extension does each subgroup correspond? Compute the conductor of each extension.

Exercise 2. Show that $\mathbb{Q}_2(\zeta_7)$ is the unique cubic extension of $\mathbb{Q}_2$.

Exercise 3. Construct explicitly the local Artin map in the case $K = \mathbb{Q}_p$.
Hint: The decomposition $K^{\mathrm{ab}} = K_\pi \cdot K^{\mathrm{un}}$ becomes in this case

$$\mathbb{Q}_p^{\mathrm{ab}} = \mathbb{Q}_p(\zeta_{p^\infty}) \cdot \left(\bigcup_n \mathbb{Q}_p(\zeta_{p^n-1}) \right).$$

Hence $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \times \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}$. So we need a suitable map

$$\mathbb{Q}_p^\times \to \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}}.$$

Exercise 4. Let p be an odd prime.

(a) Compute the norm group of $\mathbb{Q}_p(\zeta_{p^\infty})^\times$ in $\widehat{\mathbb{Q}_p^\times} = \varprojlim_L \mathbb{Q}_p^\times / \mathrm{Nm}(L^\times)$.
(b) Show that $\mathbb{Q}_p(\sqrt[p-1]{-p}) = \mathbb{Q}_p(\zeta_p)$.

2. Quadratic reciprocity and Artin reciprocity

For p an odd prime, let $p^* = (-1)^{\frac{p-1}{2}} p$. Recall that $\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}$ is the unique quadratic subextension of $\mathbb{Q}(\zeta_p)/\mathbb{Q}$, and that the image of $\mathrm{Frob}_q \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ (for $q \neq p$ an odd prime) in $\mathrm{Gal}(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q})$ acts as $\left(\frac{p^*}{q}\right)$.

Exercise 5.
(a) Prove that $\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}$ is ramified only at p if $p \equiv 1 \pmod 4$, and at p and ∞ if $p \equiv 3 \pmod 4$.
(b) Show that Artin reciprocity implies the quadratic reciprocity law

$$\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right).$$