MAT4250 EXERCISE SHEET 7

1. Group cohomology

Exercise 1. Let $L = \mathbb{Q}(i)$ and $G = \mathrm{Gal}(L/\mathbb{Q})$. Use that $H^1(G, L^\times) = 0$ to determine the rational points on the unit circle.

Exercise 2. Let $H \subseteq G$ be a subgroup of finite index.

(a) Show that $\mathrm{Cor} \circ \mathrm{Res}$ is multiplication by $(G : H)$ on $H^r(G, M)$.

(b) Prove that if G is finite and M is finitely generated as an abelian group, then $H^r(G, M)$ is finite.

Exercise 3. Read about cup products and §2: Homology in Milne's notes (pp. 73–77).

2. Kummer theory

Let K be a number field containing a primitive n-th root of unity ζ_n. Kummer theory classifies the abelian extensions of exponent n of K. (Recall that the exponent of a (finite) group G is the smallest n such that $\sigma^n = 1$ for all $\sigma \in G$.)

Let L/K be a finite Galois extension with Galois group G.

Exercise 4. Use the exact sequence

$$1 \to \mu_n \to L^\times \xrightarrow{(-)^n} L^{\times n} \to 1$$

to prove that

$$(K^\times \cap L^{\times n})/K^{\times n} \cong \mathrm{Hom}(G, \mu_n).$$

Exercise 5. Show that there is a bijection

$$\{W : K^{\times n} \subseteq W \subseteq K^\times \text{ finite index subgroup}\} \longleftrightarrow \{L/K \text{ abelian of exponent } n\}$$

$$W \longmapsto K(\sqrt[n]{W})$$

$$K^\times \cap L^{\times n} \longleftarrow\!\shortmid L$$