MAT4250 EXERCISE SHEET 8

1. Tate cohomology

Exercise 1. Let l/k be an extension of finite fields, with Galois group G. Show that $H_T^r(G, l^\times) = 0$ for all $r \in \mathbb{Z}$.

Exercise 2. Show directly, without using cohomology, that the norm map is surjective for finite fields.

Exercise 3.

(a) Compute $H_T^r(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times)$ for $r \in \mathbb{Z}$.

(b) Describe the invariant map $\mathrm{inv}_{\mathbb{C}/\mathbb{R}} \colon H^2(\mathbb{C}/\mathbb{R}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

Exercise 4. The cup product on Tate cohomology is a family of maps

$$\smile \colon H_T^p(G, M) \otimes H_T^q(G, N) \to H_T^{p+q}(G, M \otimes N),$$

where $M \otimes N$ is made into a G-module by setting $\sigma(x \otimes y) = \sigma x \otimes \sigma y$.

For $p = q = 0$, this map is the homomorphism

$$H_T^0(G, M) \otimes H_T^0(G, N) = \frac{M^G}{\mathrm{Nm}_G(M)} \otimes \frac{N^G}{\mathrm{Nm}_G(N)} \longrightarrow \frac{(M \otimes N)^G}{\mathrm{Nm}_G(M \otimes N)} = H_T^0(G, M \otimes N)$$

induced by

$$M^G \otimes N^G \to (M \otimes N)^G$$
$$x \otimes y \mapsto x \otimes y.$$

Show that this map is well defined, i.e., induces a map on H_T^0 as claimed.