MAT4250 EXERCISE SHEET 9

1. Tate cohomology

Exercise 1. If L/K is a finite unramified extension of local fields, recall that $H_T^r(\mathrm{Gal}(L/K), U_L) = 0$ for all $r \in \mathbb{Z}$.

Give an example of a ramified extension L/K such that $H_T^r(\mathrm{Gal}(L/K), U_L) \neq 0$.

2. Hilbert symbols

Exercise 2.

(a) Read the proof of Theorem 4.4 in Ch. III in Milne's notes.

Let K_v be the completion of a number field at a place v. Suppose $\mu_n \subseteq K_v$, and let $(-,-)_v$ denote the Hilbert symbol on K_v. By Remark 4.5, p. 114 in Milne's notes, the Hilbert symbol can be defined as

$$(x, y)_v = \frac{(y, K_v(\sqrt[n]{x})/K_v)(\sqrt[n]{x})}{\sqrt[n]{x}} \in \mu_n,$$

where $(-, K_v(\sqrt[n]{x})/K_v) = \phi_{K_v(\sqrt[n]{x})/K_v}$ is the local Artin map for the extension $K_v(\sqrt[n]{x})/K_v$.

(b) Show that $(x, 1-x)_v = 1$ for any $x \in K_v^\times \setminus \{1\}$. (This is referred to as the *Steinberg relation.*)

Hint: We must show that $1 - x$ is a norm from $L = K_v(\sqrt[n]{x})$. If $d = [L : K_v]$ we can write $L = K_v(\sqrt[d]{a})$ for some $a \in K_v^\times$. Show that $\mathrm{Nm}_{L/K_v}\left(\prod_{i=1}^{n/d}(1 - \zeta_n^i \sqrt[d]{a})\right) = 1 - x$.

(c) Use bimultiplicativity and the Steinberg relation to show the relations

(i) $(1, x)_v = (x, 1)_v = 1$

(ii) $(x, -x)_v = 1$

(iii) $(x, y)_v = (y, x)_v^{-1}$

(iv) $(x, x)_v = (x, -1)_v = (-1, x)_v$.

Hint: For (ii), manipulate with $1 = \left(\frac{1}{x}, 1 - \frac{1}{x}\right)_v$.

(d) Describe the Hilbert symbol $(-,-)_\infty$ on $\mathbb{Q}_\infty = \mathbb{R}$.

Exercise 3. For a field F, let $K_2(F)$ denote the free abelian group generated by symbols $\{x, y\}$ for $x, y \in F^\times$, modulo the relations

$$\{xy, z\} = \{x, z\}\{y, z\}, \quad x, y \in F^\times$$
$$\{x, yz\} = \{x, y\}\{x, z\}, \quad x, y \in F^\times$$
$$\{x, 1-x\} = 1, \quad x \in F^\times \setminus \{1\}.$$

Thus, by Exercise 2, the elements $\{x, y\}$ of $K_2(F)$ automatically satisfy relations (i)–(iv) above.

(a) If F is a finite field, show that $K_2(F) = 1$.

A *symbol* (also called a *Steinberg symbol*) on a field F, with values in an abelian group A (written multiplicatively), is a bimultiplicative map

$$(-,-)\colon F^\times \times F^\times \to A$$

satisfying the Steinberg relation, i.e., $(x, 1-x) = 1$. Thus, by construction, the group $K_2(F)$ is the universal object with respect to symbols on F, in the sense that any symbol on F factors uniquely through $K_2(F)$.

(b) Let F be a field with a discrete valuation v. Let $k(v) = \mathcal{O}_v/\mathfrak{p}_v$ denote the corresponding residue field. Show that the map

$$\partial_v \colon F^\times \times F^\times \to k(v)^\times$$

defined by

$$\partial_v(x,y) = (-1)^{\mathrm{ord}_v(x)\mathrm{ord}_v(y)} x^{\mathrm{ord}_v(y)} y^{-\mathrm{ord}_v(x)} \pmod{\mathfrak{p}_v}$$

is a symbol on F. It is called the *tame symbol*.

(c) For each prime p we have a corresponding tame symbol ∂_p on $\mathbb{Q}$. Collecting all these we obtain a homomorphism

$$\partial \colon K_2(\mathbb{Q}) \to \bigoplus_p \mathbb{F}_p^\times.$$

Is ∂ surjective? Can you find an element in the kernel of ∂?

3. The Kronecker–Weber theorem

Exercise 4.

(a) Show that the norm group for the extension $\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p$ is $p^{\mathbb{Z}} \oplus U_p^n$.

(b) Use the result of (a) to prove the *local Kronecker–Weber theorem*: Every finite abelian extension L of $\mathbb{Q}_p$ is contained in $\mathbb{Q}_p(\zeta)$ for some root of unity ζ.

We will now use the local Kronecker–Weber theorem to prove the global Kronecker–Weber theorem:

Theorem (Kronecker–Weber)**.** *Every finite abelian extension L of $\mathbb{Q}$ is contained in $\mathbb{Q}(\zeta)$ for some root of unity ζ.*

Exercise 5. Let $L/\mathbb{Q}$ be a finite abelian extension, and let S be the set of ramified finite places. Let $L_v/\mathbb{Q}_p$ denote the completion of L at some place v of L above $p \in S$. Then $L_v \subseteq \mathbb{Q}_p(\mu_{n_p})$ for some n_p by the local Kronecker–Weber theorem. Let $e_p = \mathrm{ord}_p(n_p)$ and set $n = \prod_{p \in S} p^{e_p}$. We aim to show that $L \subseteq \mathbb{Q}(\mu_n)$.

(a) Let $M = L(\mu_n)$, and let w be a place of M above v. Show that M_w is the compositum

$$M_w = \mathbb{Q}_p(\mu_{p^{e_p}})\mathbb{Q}_p(\mu_m)$$

for some m such that $p \nmid m$.

(b) Let I_p be the inertia group of $M_w/\mathbb{Q}_p$. Show that

$$I_p \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^{e_p}})/\mathbb{Q}_p).$$

(c) Let I be the subgroup of $\mathrm{Gal}(M/\mathbb{Q})$ generated by I_p for all $p \in S$. Show that $I = \mathrm{Gal}(M/\mathbb{Q})$.

(d) Show that $[M : \mathbb{Q}] = [\mathbb{Q}(\mu_n) : \mathbb{Q}]$ and conclude that $L \subseteq \mathbb{Q}(\mu_n)$.