

Exercise 1

(1)

$$a) K = \mathbb{Q}(\sqrt{-5}) \Rightarrow C = \mathcal{U}(K) = \mathbb{Z}/2.$$

$$m = (2) = (2, 1 + \sqrt{-5})^2$$

We use the exact sequence

$$1 \rightarrow \mathcal{U}/\mathcal{U}_{m,1} \rightarrow K_m/K_{m,1} \rightarrow C_m \rightarrow C \rightarrow 1$$

$$\text{Here } \mathcal{O}_K = \mathbb{Z}[\sqrt{-5}], \quad \mathcal{U} = \{\pm 1\},$$

$$K_m/K_{m,1} \cong (\mathcal{O}_K/m_0)^* \cong (\mathbb{Z}/2)^* = 1$$

and since both 1 and -1 map to 1 in $(\mathbb{Z}/2)^*$

$\mathcal{U}_{m,1} = \mathcal{U}_m$. Hence the exact sequence becomes

$$1 \rightarrow C_m \xrightarrow{\cong} \mathbb{Z}/2 \rightarrow 1 \quad \text{so } C_m \cong \mathbb{Z}/2.$$

2) b) For $K = \mathbb{Q}(\sqrt{3})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$, $\mathcal{O}_K^* = U = \pm(2+\sqrt{3})^{\mathbb{Z}} \cong \mu_2 \oplus \mathbb{Z}$,
 $C = \mathcal{O}(K) = 1$.

Now $2+\sqrt{3}$ is positive both at ∞ , and ∞_c ,

since $2 \pm \sqrt{3} > 0$. Thus $U_{m,1} = (2+\sqrt{3})^{\mathbb{Z}}$

Hence the exact sequence becomes

$$1 \rightarrow \mu_2 \rightarrow \mu_2^2 \rightarrow C_m \rightarrow 1,$$

giving $C_m \cong \mathbb{Z}/2$.

Exercise 2. Note that, for an abelian extension L/K ,

an unramified prime ideal \mathfrak{p} of K lies
in $\ker(\psi_{L/K}) \iff \mathfrak{p}$ splits completely in L/K .

This is because the Frobenius $(\mathfrak{p}, L/K) = \psi_{L/K}(\mathfrak{p})$
acts by raising elements in the residue field to
 $f(\mathfrak{p}/\mathfrak{p})^{\text{th}}$ power (for $\mathfrak{p}/\mathfrak{p}$), which is 1 $\iff \mathfrak{p}$
splits completely (since \mathfrak{p} is unramified).

So, finding the smallest m s.t. $i(\mathbb{Q}_{m,1}) \subseteq \ker(\psi_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}})$

amounts to finding $\text{Spl}_S(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$,

done on p. 3 in Milne using quadratic
reciprocity.

Exercise 3. See Solution to A-3, p. 270
in Milne.

Exercise 4.

$$\begin{aligned} \text{Let } I(S) &= \langle \mathfrak{p} : \mathfrak{p} \in S \rangle \subseteq I_K \\ &\cong \bigoplus_{\mathfrak{p} \in S} \mathbb{Z} \end{aligned}$$

We have an exact sequence

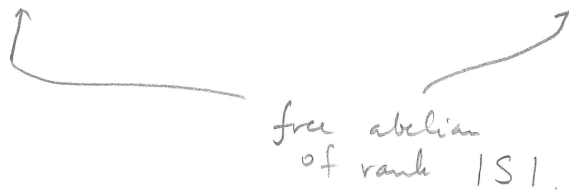
$$\begin{aligned} 1 \rightarrow U_K \rightarrow (\mathcal{O}_K^S)^* \xrightarrow{i} I(S) \\ \parallel \\ \mathcal{O}_K^* \cong \mu(K) \oplus \mathbb{Z}^{r+s-1} \end{aligned}$$

where $i(a) = (a)$.

Let $h = |C_K|$ be the class number of K .

Then $I(S)^h \subseteq \text{im}(i)$, since if $\mathfrak{p} \in I(S)$, \mathfrak{p}^h is principal and in the image of i .

$$\text{So } I(S)^h \subseteq i((\mathcal{O}_K^S)^*) \subseteq I(S)$$



free abelian
of rank $|S|$.

Hence $i((\mathcal{O}_K^S)^*)$ is free of rank $|S|$.

It follows that the exact sequence $1 \rightarrow U_K \rightarrow (\mathcal{O}_K^S)^* \rightarrow \text{im}(i) \rightarrow 1$

splits, giving

$$\begin{aligned} (\mathcal{O}_K^S)^* &\cong U_K \times \text{im}(i) \\ &\cong \mu(K) \times \mathbb{Z}^{r+s-1} \times \mathbb{Z}^{|S|} \end{aligned}$$

