Exercise 1

a) $K = \mathbb{Q}(\sqrt{-5}) \Rightarrow C = Cl(K) = \mathbb{Z}/2$.

$m = (2) = (2, 1+\sqrt{-5})^2$

We use the exact sequence

$$1 \to U/U_{m,1} \to K_m/K_{m,1} \to C_m \to C \to 1$$

Here $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$, $U = \{\pm 1\}$,

$$K_m/K_{m,1} \cong (\mathcal{O}_K/m_0)^* \cong (\mathbb{Z}/2)^* = 1$$

and since both 1 and -1 map to 1 in $(\mathbb{Z}/2)^*$

$U_{m,1} = U_m$. Hence the exact sequence becomes

$$1 \to C_m \xrightarrow{\cong} \mathbb{Z}/2 \to 1 \quad \text{so} \quad C_m \cong \mathbb{Z}/2.$$

2) b) For $K = \mathbb{Q}(\sqrt{3})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{3}]$, $\mathcal{O}_K^* = U = \pm(2+\sqrt{3})^{\mathbb{Z}} \cong \mu_2 \oplus \mathbb{Z}$,
$C = Cl(K) = 1$.

Now $2+\sqrt{3}$ is positive both at ∞_1 and ∞_2, since $2 \pm \sqrt{3} > 0$. Thus $U_{m,1} = (2+\sqrt{3})^{\mathbb{Z}}$

Hence the exact sequence becomes

$$1 \to \mu_2 \to \mu_2^2 \to C_m \to 1,$$

giving $C_m \cong \mathbb{Z}/2$.

Exercise 2. Note that, for an abelian extension L/K, an unramified prime ideal $\mathfrak{p}$ of K lies in $\ker(\psi_{L/K}) \iff \mathfrak{p}$ splits completely in L/K.

This is because the Frobenius $(\mathfrak{p}, L/K) = \psi_{L/K}(\mathfrak{p})$ acts by raising elements in the residue field to $f(\mathfrak{q}/\mathfrak{p})^{th}$ power (for $\mathfrak{q}/\mathfrak{p}$), which is $1 \iff \mathfrak{p}$ splits completely (since $\mathfrak{p}$ is unramified).

So, finding the smallest m s.t. $i(\mathbb{Q}_{m,1}) \subseteq \ker(\psi_{\mathbb{Q}(\sqrt{a})/\mathbb{Q}})$ amounts to finding $Spl_S(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$, done on p. 3 in Milne using quadratic reciprocity.

Exercise 3. See Solution to A-3, p. 270 in Milne.

Exercise 4.

Let $I(S) = \langle \mathfrak{p} : \mathfrak{p} \in S \rangle \subseteq I_K$

$\cong \bigoplus_{\mathfrak{p} \in S} \mathbb{Z}$

We have an exact sequence

$$1 \to U_K \to (\mathcal{O}_K^S)^* \xrightarrow{i} I(S)$$

$U_K = \mathcal{O}_K^* \cong \mu(K) \oplus \mathbb{Z}^{r+s-1}$

where $i(a) = (a)$.

Let $h = |C_K|$ be the class number of K.

Then $I(S)^h \subseteq im(i)$, since if $\mathfrak{p} \in I(S)$, $\mathfrak{p}^h$ is principal and in the image of i.

So $I(S)^h \subseteq i((\mathcal{O}_K^S)^*) \subseteq I(S)$

free abelian of rank $|S|$.

Hence $i((\mathcal{O}_K^S)^*)$ is free of rank $|S|$.

It follows that the exact sequence $1 \to U_K \to (\mathcal{O}_K^S)^* \to im(i) \to 1$ splits, giving

$$(\mathcal{O}_K^S)^* \cong U_K \times im(i).$$

$U_K \cong \mu(K) \times \mathbb{Z}^{r+s-1}$, $im(i) \cong \mathbb{Z}^{|S|}$