Exercise 1

If $\underline{a} \in \mathbb{I}_K$, then $\underline{a}$ has a basic open neighborhood $\underline{a} \in \prod_{v \in S} V_v \times \prod_{v \notin S} \mathcal{O}_v^*$, $V_v \subseteq K_v^*$ open, such that V_v are bounded $\forall\, v \in S_\infty$.

Then the closure of each V_v is compact in K_v^*, and since $\mathcal{O}_v^*$ is compact, it follows that $\prod_{v \in S} \overline{V_v} \times \prod_{v \notin S} \mathcal{O}_v^*$ is compact, by Tychonoff.

Exercise 2

Each K_v^* is Hausdorff, so $\mathbb{I}_K$ is Hausdorff. Since K^* is discrete in $\mathbb{I}_K$, C_K is also Hausdorff.

Exercise 3

For $(x,y) \subseteq \mathbb{R}_{>0}$, we must show that $c^{-1}((x,y))$ is open. Let $\underline{a} = (a_v)_v \in \mathbb{I}_K$ s.t. $c(\underline{a}) \in (x,y)$. Let w be a real place of K, and let $S = \{ v \mid a_v \notin \mathcal{O}_v^* \}$.

Put

$$U_\varepsilon = \prod_{\substack{v \in S \\ v \neq w}} B(a_v, 1) \times \prod_{v \notin S} \mathcal{O}_v^* \times B(a_w, \varepsilon) \ni \underline{a}$$

Then U_ε is open in $\mathbb{I}_K$, and for sufficiently small ε we have $c(U_\varepsilon) \subseteq (x,y)$.

2)

Exercise 4

The sequence $1 \to \mathbb{I}_K^1 \to \mathbb{I}_K \xrightarrow{c} \mathbb{R}_{>0} \to 1$

is split by the continuous map

$$\begin{aligned} \mathbb{R}_{>0} &\longrightarrow \mathbb{I}_K \\ t &\longmapsto (a_v)_v \quad \text{where } a_v = \begin{cases} 1, & v \text{ finite} \\ t^{\frac{1}{n}} & v \text{ infinite}, \end{cases} \end{aligned}$$

where $n = [K:\mathbb{Q}]$.

This implies $\mathbb{I}_K \cong \mathbb{I}_K^1 \times \mathbb{R}_{>0}$ and similarly for C_K.

Exercise 5

a) $\mathbb{R}^* \cong \{\pm 1\} \times \mathbb{R}_{>0}$, hence

$$\begin{aligned} \mathbb{I}_\mathbb{Q} &\cong \{\pm 1\} \times \mathbb{R}_{>0} \times \prod_p{}' (\mathbb{Z} \times \mathbb{Z}_p^*) \\ &\cong \{\pm 1\} \times \mathbb{R}_{>0} \times \bigoplus_p \mathbb{Z} \times \prod_p \mathbb{Z}_p^*, \end{aligned}$$

where the restricted product demands $\oplus$ for the $\mathbb{Z}$-copies.

b) $a \mapsto (\mathrm{sgn}(a), \mathrm{ord}_2(a), \mathrm{ord}_3(a), \dots, \mathrm{ord}_p(a), \dots)$
gives an isomorphism $\mathbb{Q}^* \longrightarrow \{\pm 1\} \times \bigoplus_p \mathbb{Z}$.
Hence $\mathbb{I}_\mathbb{Q} \cong \mathbb{Q}^* \times \mathbb{R}_{>0} \times \hat{\mathbb{Z}}^*$.

c) Since every abelian finite ext. of $\mathbb{Q}$ is contained in a cyclotomic field, $\mathbb{Q}^{ab} = \bigcup_n \mathbb{Q}(\zeta_n)$, and
$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \varprojlim_n \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong \varprojlim (\mathbb{Z}/n)^* = \hat{\mathbb{Z}}^* \cong C_\mathbb{Q}^1$.