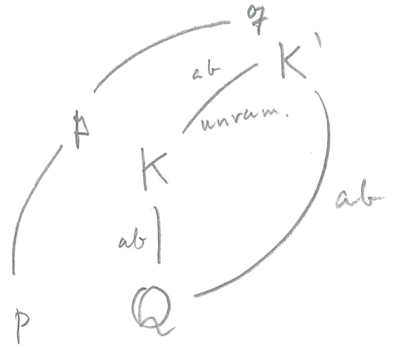
Exercise 1

See end of p. 162 in Milne.

Exercise 2

Given a rational prime p and

(a) $\mathfrak{q} \mid \mathfrak{p} \mid p$:



Since K'/K is unramified,

$$Nm(U_{\mathfrak{q}}) = U_{\mathfrak{p}} \subseteq K_{\mathfrak{p}}^{*}.$$

Hence

$$U_p^f \subseteq Nm(K_{\mathfrak{p}}^{*}) \Leftrightarrow U_p^f \subseteq Nm((K'_{\mathfrak{q}})^{*})$$

Hence the local conductors are equal:

$f(K_{\mathfrak{p}}/\mathbb{Q}_p) = f(K'_{\mathfrak{q}}/\mathbb{Q}_p)$, hence the result follows, the global conductor being the product of the local conductors.

(b) $f(\mathbb{Q}(i,\sqrt{-5})/\mathbb{Q}) = f(\mathbb{Q}(\sqrt{-5})/\mathbb{Q})$ since $\mathbb{Q}(i,\sqrt{-5})$ is the Hilbert class field of $\mathbb{Q}(\sqrt{-5})$ and $\mathbb{Q}(i,\sqrt{-5})$ is abelian over $\mathbb{Q}$.

And we have $f(\mathbb{Q}(\sqrt{-5})/\mathbb{Q}) = (20)\infty$.

2)

Exercise 3

$$\begin{tikzcd}[column sep=small] & K \arrow[dl, dash] \arrow[dd, dash, bend left] \\ \mathbb{Q}(\zeta_p) \arrow[dr, dash] & \\ & \mathbb{Q} \end{tikzcd}$$

If $K/\mathbb{Q}$ is abelian, then, by Exercise 2,

$$f(K/\mathbb{Q}) = f(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = (p)\infty,$$

hence $K \subseteq \mathbb{Q}(\zeta_p)$, hence $K = \mathbb{Q}(\zeta_p)$, i.e., $\mathbb{Q}(\zeta_p)$ is its own Hilbert class field.

That means $Cl(\mathbb{Q}(\zeta_p)) \cong Gal(K/\mathbb{Q}(\zeta_p)) = \{1\}$, i.e., the class group of $\mathbb{Q}(\zeta_p)$ is trivial.

(This happens only for a few values of p $(2 \leq p \leq 19)$, in fact the class number of $\mathbb{Q}(\zeta_p)$ grows exponentially as p increases. Thus the Hilbert class field of $\mathbb{Q}(\zeta_p)$ is almost never abelian over $\mathbb{Q}$).

Exercise 4

$$L(1,\chi) = \sum_{n \geq 1} \frac{\chi(n)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

Exercise 5

(a) First note that $K_2(\mathbb{Q})$ is generated by $\{a,b\}$ for $a,b \in \mathbb{Z} \setminus \{0\}$. Indeed, by bilinearity,
$\{\frac{x}{y}, z\} = \{x,z\}\{y^{-1}, z\} = \{x,z\}\{z,y\}$.

We can write any symbol $\{a,b\}$ $(a,b \in \mathbb{Z} \setminus \{0\})$

as $\underbrace{\{a,b\} = \{|a|,|b|\}}_{\in \Lambda_0} \cdot \xi$, $\quad \xi \in \langle \{-1,-1\}, \{c,-1\} \rangle$

(by expanding $\{\pm a, \pm b\}$ using bimultiplicativity)

Since $\{c,-1\} = \{c,c\}$, any $\{a,b\} \in K_2(\mathbb{Q})$ lies in $\Lambda_0 \oplus \Lambda_\infty$.

We note that $\Lambda_0 \cap \Lambda_\infty = 1$, since $(-1,-1)_\infty = -1$ while $(|a|,|b|)_\infty = 1$, so the sum is a direct sum.

6) (b) $\{2,2\} = \{2,-1\} = \{2, 1-2\} = 1$ by the Steinberg relation.

So $\Lambda_2 \cong \mathbb{F}_2^*$.

(c) We show that $\varphi \circ \psi = id$ and $\psi \circ \varphi = id$; from the proof of this it will become clear that ψ is well defined and multiplicative.

- For $\varphi \circ \psi$ we have $\varphi \circ \psi(\bar{x}) = \partial_p(\{x, p\}) = \bar{x}$ by the formula for the tame symbol.

- For $\psi \circ \varphi$ we have $\psi(\varphi(\{a,b\})) = \{c, p\}$, where $\bar{c} = (-1)^{v_p(a)v_p(b)} a^{v_p(b)} b^{-v_p(a)} \bmod p$; we need to show $\{c,p\} \equiv \{a,b\} \bmod \Lambda_q$ when we pick c s.t. $1 \leq c < p$. We check the cases:

(i) $1 \leq a, b < p \Rightarrow \bar{c} = \bar{1}$ and $\{a,b\} \in \Lambda_q$, so ok.

(ii) $a < p$, $b = p \Rightarrow \bar{c} = \bar{a}$ and $\{c, p\} = \{a, p\} = \{a, b\}$.

(iii) $a = p$, $b < p$. Then $\bar{c} = \bar{b}^{-1}$, so $cb \equiv 1 \bmod p$ with $1 \leq c < p$.

We must show $\{p, bc\} \equiv 1 \bmod \Lambda_q$.

Write $bc = 1 + dp$, where we may assume $b > 1$, $0 < d < p$.

Then

$$1 = \{-dp, 1+dp\}$$
$$= \{-dp, bc\}$$
$$= \underbrace{\{-d, b\}\{-d, c\}}_{\in \Lambda_q}, \{p, bc\}.$$

$\in \Lambda_q$ using $\{-d, b\} = \{-1, b\}\{d, b\}$
$= \{b, b\}\{d, b\}$
& similarly for $\{-d, c\}$.

(iv) $a = p = b \Rightarrow \bar{c} = -1$ & $c = p-1$,

So $\psi(\bar{c}) = \{p-1, p\}$

$$= \{-(1-p), p\}$$
$$= \{-1, p\}\underbrace{\{1-p, p\}}_{=1}$$
$$= \{-1, p\} = \{p, p\}.$$