Exercise 1

a) The norm $N_{L/K}$ is homogeneous of degree $n = [L:K]$, so

$$N_{L/K}(s^{-1}x) = s^{-n} N_{L/K}(x).$$

Hence $Nm(S^{-1}\mathfrak{a}) \subseteq S^{-1} Nm(\mathfrak{a})$.

For the other inclusion, we have

$$s^{-1} N_{L/K}(x) = s^{n-1} N_{L/K}(s^{-1}x).$$

b) Clearly $Nm(\mathfrak{a}\mathfrak{b}) \subseteq Nm(\mathfrak{a}) Nm(\mathfrak{b})$, and by multiplicativity of $N_{L/K}$, equality holds if $\mathfrak{a}$ & $\mathfrak{b}$ are principal.

Thus, since $A_{\mathfrak{p}}$ is a DVR $\forall$ primes $\mathfrak{p}$ of A, we have equality $Nm(\mathfrak{a}_{\mathfrak{p}} \cdot \mathfrak{b}_{\mathfrak{p}}) = Nm(\mathfrak{a}_{\mathfrak{p}}) Nm(\mathfrak{b}_{\mathfrak{p}})$ for all $\mathfrak{p}$, hence this holds globally as well.

2) c) Localizing at a prime $\mathfrak{p}'$ of A we have $Nm(\mathfrak{q})_{\mathfrak{p}'} = Nm(\mathfrak{q}_{\mathfrak{p}'})$, and $\mathfrak{q}_{\mathfrak{p}'} = B_{\mathfrak{p}'}$ unless $\mathfrak{p}' = \mathfrak{p}$. Hence $Nm(\mathfrak{q})$ is a power of $\mathfrak{p}$, and we need to show it is $f = f(\mathfrak{q}/\mathfrak{p})$.

Localizing again at $\mathfrak{p}$, we may assume $A = A_{\mathfrak{p}}$ is a DVR with max. ideal (p), and then that B is a semilocal PID with $\mathfrak{q} = (\pi)$.

So we need to show $N_{L/K}(\pi) = u p^f$ for u a unit. This is established by computing $\mathrm{ord}_p(\det \pi)$ using induction on the rank of B as an A-module.

Exercise 2

- For $\mathfrak{p} = (2, 1+\sqrt{-5})$:

We know $\mathfrak{p}^2 = (2)$, so $\mathfrak{p}$ lies above 2 and $Nm(\mathfrak{p}) = (2)^f = (2)$.

- For $\mathfrak{p} = (3, 1+\sqrt{-5})$:

$$Nm(\mathfrak{p}) = \left(N_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(3),\ N_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(1+\sqrt{-5}) \right)$$
$$= (9, 6) = (3).$$

Exercise 3

By Furtwängler's theorem, any finite unramified extension arises as the class field of some $H \subseteq Cl(\mathbb{Q}(\sqrt{-1}))$. Since $Cl(\mathbb{Q}(\sqrt{-1})) = 0$, the only such is the trivial extension.

Exercise 4

$$\begin{tikzcd} & K(i) \arrow[dd, dash] \\ K=\mathbb{Q}(\sqrt{-5}) \arrow[ur, dash] \arrow[dr, dash] & \\ & \mathbb{Q} \end{tikzcd}$$

If L is the Hilbert class field of $K = \mathbb{Q}(\sqrt{-5})$ we know that
$$Gal(L/K) \cong Cl(K) = \mathbb{Z}/2,$$
hence $[L:K] = 2$.

Now $K(i)/K$ is at most ramified over 2 (by multiplicativity of ramification indices in towers). But $\mathbb{Q}(\sqrt{-5}, i) = \mathbb{Q}(\sqrt{-5}, \sqrt{5})$, and therefore 2 does not ramify in the extension $K(i)/K$.
Finally, no infinite places ramify, so $K(i)$ is the Hilbert class field.

4)

Exercise 5

- For $\mathfrak{m} = (5)$: We can list $I^{S(\mathfrak{m})}$ and

$$P^{\mathfrak{m}} = \left\langle (a) \;\middle|\; \begin{array}{l} \mathrm{ord}_{\mathfrak{p}}(a-1) \geq \mathrm{ord}_{\mathfrak{p}}(\mathfrak{m}_0) \;\; \forall\, \mathfrak{p} \mid \mathfrak{m}_0 \\ \sigma_v(a) > 0 \;\; \forall\, v \mid \mathfrak{m}_\infty,\; \sigma_v : K \hookrightarrow \mathbb{R} \end{array} \right\rangle$$

as follows:

$$I^{S(\mathfrak{m})} = \{(1), (\tfrac{1}{2}), (2), (\tfrac{1}{3}), (\tfrac{2}{3}), (\tfrac{1}{4}), (\tfrac{3}{4}), (4), (\tfrac{1}{6}), (6), \dots\}$$

For $P^{\mathfrak{m}}$, note that e.g. $(\tfrac{2}{3}) \in P^{\mathfrak{m}}$ since

$(\tfrac{2}{3}) = (-\tfrac{2}{3})$ and $\mathrm{ord}_5(-\tfrac{2}{3} - 1) = \mathrm{ord}_5(5) = 1$.

We find:

$$P^{\mathfrak{m}} = \{(1), (\tfrac{2}{3}), (\tfrac{3}{2}), (\tfrac{1}{4}), (4), (6), (\tfrac{1}{6}), (\tfrac{2}{7}), (\tfrac{7}{2}), \dots\}$$

Hence

$$C_{\mathfrak{m}} = I^{S(\mathfrak{m})} / P^{\mathfrak{m}} = \{[(1)], [(2)]\}$$

$$\cong (\mathbb{Z}/5)^* / \{\pm 1\}$$

- For $\mathfrak{m} = (5)\infty$ we get

$$P^{\mathfrak{m}} = \{(1), (6), (\tfrac{1}{6}), (\tfrac{2}{7}), (\tfrac{7}{2}), \dots\}$$

and $C_{\mathfrak{m}} \cong (\mathbb{Z}/5)^*$

Exercise 6. See p. 6 in Milne's CFT notes.