

## Exercise 1

(1)

a) The norm  $N_{L/K}$  is homogeneous of degree  $n = [L:K]$ , so

$$N_{L/K}(s \cdot x) = s^n N_{L/K}(x).$$

Hence  $Nm(S^{-1}\alpha) \subseteq S^{-1}Nm(\alpha)$ .

For the other inclusion, we have

$$s^{-1}N_{L/K}(x) = s^{n-1}N_{L/K}(s^{-1}x).$$

b) Clearly  $Nm(\alpha\beta) \subseteq Nm(\alpha)Nm(\beta)$ ,

and by multiplicativity of  $N_{L/K}$ , equality holds if  $\alpha$  &  $\beta$  are principal.

Thus, since  $A_{\mathfrak{p}}$  is a DVR  $\forall$  primes  $\mathfrak{p}$  of  $A$ , we have equality  $Nm(\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}}) = Nm(\alpha_{\mathfrak{p}})Nm(\beta_{\mathfrak{p}})$  for all  $\mathfrak{p}$ , hence this holds globally as well.

2) c) Localizing at a prime  $\mathfrak{p}'$  of  $A$  we have  $Nm(\mathfrak{o}_f)_{\mathfrak{p}'} = Nm(\mathfrak{o}_{f, \mathfrak{p}'})$ , and  $\mathfrak{o}_{f, \mathfrak{p}'} = B_{\mathfrak{p}'}$

unless  $\mathfrak{p}' = \mathfrak{p}$ . Hence  $Nm(\mathfrak{o}_f)$  is a power of  $\mathfrak{p}$ , and we need to show it is  $f = f(\mathfrak{o}_f/\mathfrak{p})$ .

Localizing again at  $\mathfrak{p}$ , we may assume  $A = A_{\mathfrak{p}}$  is a DVR with max. ideal  $(\mathfrak{p})$ ,

and then that  $B$  is a semilocal PID with  $\mathfrak{o}_f = (\pi)$ .

So we need to show  $N_{L/K}(\pi) = u \mathfrak{p}^f$  for  $u$  a unit. This is established by computing  $\text{ord}_{\mathfrak{p}}(\det \pi)$  using induction on the rank of  $B$  as an  $A$ -module.

## Exercise 2

3

• For  $\mathfrak{p} = (2, 1 + \sqrt{-5})$ :

We know  $\mathfrak{p}^2 = (2)$ , so  $\mathfrak{p}$  lies above 2 and

$$N_m(\mathfrak{p}) = (2)^f = (2).$$

• For  $\mathfrak{p} = (3, 1 + \sqrt{-5})$ :

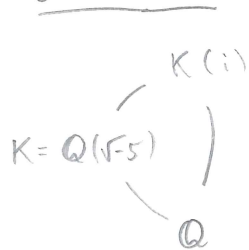
$$N_m(\mathfrak{p}) = (N_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(3), N_{\mathbb{Q}(\sqrt{-5})/\mathbb{Q}}(1 + \sqrt{-5}))$$

$$= (9, 6) = (3).$$

## Exercise 3

By Furtwangler's theorem, any finite unramified extension arises as the class field of some  $H \subseteq \mathcal{O}(\mathbb{Q}(\sqrt{-1}))$ . Since  $\mathcal{O}(\mathbb{Q}(\sqrt{-1})) = 0$ , the only such is the trivial extension.

## Exercise 4



If  $L$  is the Hilbert class field

of  $K = \mathbb{Q}(\sqrt{-5})$  we know that

$$\text{Gal}(L/K) \cong \mathcal{O}(K) = \mathbb{Z}/2,$$

hence  $[L:K] = 2$ .

Now  $K(i)/K$  is at most ramified over 2 (by multiplicativity of ramification indices in towers).

But  $\mathbb{Q}(\sqrt{-5}, i) = \mathbb{Q}(\sqrt{-5}, \sqrt{5})$ , and therefore 2

does not ramify in the extension  $K(i)/K$ .

Finally, no infinite places ramify, so  $K(i)$  is the Hilbert class field.

4) Exercise 5

• For  $m = (5)$ : We can list  $I^{S(m)}$  and

$$P^m = \left\langle (a) \mid \begin{array}{l} \text{ord}_p(a-1) \geq \text{ord}_p(m_0) \quad \forall \phi \mid m_0 \\ \text{or } (a) > 0 \quad \forall \nu \mid m_\infty, \quad \sigma_\nu: K \hookrightarrow \mathbb{R} \end{array} \right\rangle$$

as follows:

$$I^{S(m)} = \left\{ (1), \left(\frac{1}{2}\right), (2), \left(\frac{1}{3}\right), \left(\frac{2}{3}\right), \left(\frac{1}{4}\right), \left(\frac{3}{4}\right), (4), \left(\frac{1}{6}\right), (6), \dots \right\}$$

For  $P^m$ , note that e.g.  $\left(\frac{2}{3}\right) \in P^m$  since

$$\left(\frac{2}{3}\right) = \left(-\frac{2}{3}\right) \text{ and } \text{ord}_5\left(-\frac{2}{3}-1\right) = \text{ord}_5(5) = 1.$$

We find:

$$P^m = \left\{ (1), \left(\frac{2}{3}\right), \left(\frac{3}{2}\right), \left(\frac{1}{4}\right), (4), (6), \left(\frac{1}{6}\right), \left(\frac{2}{7}\right), \left(\frac{7}{2}\right), \dots \right\}$$

Hence

$$C_m = I^{S(m)} / P^m = \{ [(1)], [(2)] \}$$

$$\cong (\mathbb{Z}/5)^* / \{ \pm 1 \}$$

• For  $m = (5)^\infty$  we get

$$P^m = \left\{ (1), (6), \left(\frac{1}{6}\right), \left(\frac{2}{7}\right), \left(\frac{7}{2}\right), \dots \right\}$$

and  $C_m \cong (\mathbb{Z}/5)^*$

Exercise 6. See p. 6 in Milnes' CFT notes.