Exercise 1 Recall that $\mathbb{Q}_2^* / \mathbb{Q}_2^{*2} = \{\pm 1, \pm 2, \pm 5, \pm 10\}$,

So the quadratic extensions of $\mathbb{Q}_2$ are $\mathbb{Q}_2(\sqrt{x})$ for $x = -1, \pm 2, \pm 5, \pm 10$.

On the other hand, $\mathbb{Q}_2^* = \langle 2 \rangle \oplus \langle -1 \rangle \oplus (1 + 4\mathbb{Z}_2)$.

We find the subgroups of index 2, the corresponding extension, and the conductor:

Norm subgroup	Extension	Conductor
$\langle 4 \rangle \oplus \langle -1 \rangle \oplus \langle 5 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{5})$	0
$\langle 2 \rangle \oplus \langle 5 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{-1})$	2
$\langle -2 \rangle \oplus \langle 5 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{-5})$	2
$\langle 2 \rangle \oplus \langle -1 \rangle \oplus \langle 5^2 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{2})$	3
$\langle 2 \rangle \oplus \langle -5 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{-2})$	3
$\langle 10 \rangle \oplus \langle -1 \rangle \oplus \langle 5^2 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{10})$	3
$\langle -2 \rangle \oplus \langle -5 \rangle_{\mathbb{Z}_2}$	$\mathbb{Q}_2(\sqrt{-10})$	3

2)

Exercise 2

$\mathbb{Q}_2^* \cong \mathbb{Z} \oplus \mu_2 \oplus (1+4\mathbb{Z}_2)$ has exactly one subgroup

of index 3, namely $3\mathbb{Z} \oplus \mu_2 \oplus (1+4\mathbb{Z}_2)$.

So, by local CFT, this corresponds to

a unique degree 3 extension of $\mathbb{Q}_2$.

Now $Gal(\mathbb{Q}_2(\zeta_{2^3-1})/\mathbb{Q}_2) \cong \mathbb{Z}/3$,

So $\mathbb{Q}_2(\zeta_7)$ is this unique degree 3 extension

Exercise 3

We define the map

$$\varphi_{\mathbb{Q}_p} : \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}_p^* \times \mathbb{Z}}}{\mathbb{Q}_p^*} \longrightarrow Gal(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) \cong \mathbb{Z}_p^* \times \hat{\mathbb{Z}}$$

by letting $\varphi_{\mathbb{Q}_p}$ be the identity on the

factor $\mathbb{Z}_p^*$ of $\mathbb{Q}_p^* \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^*$, and on the factor $p^{\mathbb{Z}}$,

p is mapped to the generator $1 \in \hat{\mathbb{Z}}$ (corresponding to

the Frobenius, i.e., the topological generator of $\underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \hat{\mathbb{Z}}}}{Gal(\mathbb{Q}_p^{un}/\mathbb{Q}_p)}$)

Note: The reciprocity map is not surjective, but has dense image.

Exercise 4

(a) We have $\widehat{\mathbb{Q}_p^*} \cong p^{\hat{\mathbb{Z}}} \oplus \mathbb{Z}_p^* \xrightarrow[\varphi]{\cong} \mathrm{Gal}(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) = \hat{\mathbb{Z}} \oplus \mathbb{Z}_p^*$

and $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^*$.

Hence $\widehat{\mathbb{Q}_p^*} \big/ \mathrm{Nm}(\mathbb{Q}_p(\zeta_{p^\infty})^*) \cong \mathbb{Z}_p^*$, so that

$\mathrm{Nm}(\mathbb{Q}_p(\zeta_{p^\infty})) = p^{\hat{\mathbb{Z}}}$

(b) Let $L = \mathbb{Q}_p(\sqrt[p-1]{-p})$. The minimal polynomial of $\sqrt[p-1]{-p}$ over $\mathbb{Q}_p$ is $X^{p-1} + p$, hence $\mathrm{Nm}(\sqrt[p-1]{-p}) = p$

Hence $\mathrm{Nm}(L^*) \supseteq \mathrm{Nm}(\mathbb{Q}_p(\zeta_{p^\infty})^*) = \hat{\mathbb{Z}} \subseteq \widehat{\mathbb{Q}_p^*}$,

and by the local CFT correspondence we have $L \subseteq \mathbb{Q}_p(\zeta_{p^\infty}) = \bigcup_n \mathbb{Q}_p(\zeta_{p^n})$

It follows that $L = \mathbb{Q}_p(\zeta_p)$.

Exercise 5

(a) If $p \equiv 1\ (4)$ then $p^* = p$ and $\mathcal{O}_{\mathbb{Q}(\sqrt{p^*})} = \mathbb{Z}\left[\frac{1+\sqrt{p}}{2}\right]$;

the minimal polynomial of $\frac{1+\sqrt{p}}{2}$ being

$X^2 - X + \frac{1-p}{4}$, with discriminant $\Delta = p$. So,

since $\mathbb{Q}(\sqrt{p})$ is real, only p ramifies.

If $p \equiv 3\ (4)$, then $p^* = -p \equiv 1\ (4)$ and

the same story as above goes, except that

$\mathbb{Q}(\sqrt{p^*})$ is now nonreal and ∞ ramifies as well.

(b) Let $\mathfrak{m} = (p)\infty$ The Artin map for $\mathfrak{m}$ is

$$\begin{aligned} I^{S(\mathfrak{m})} &\longrightarrow \mathrm{Gal}\left(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}\right) \\ (q) &\longmapsto \mathrm{Frob}_q = \left(\frac{p^*}{q}\right) \end{aligned}$$

Now Artin reciprocity gives a surjection

$$\begin{array}{c} C_{\mathfrak{m}} = \dfrac{I^{S(\mathfrak{m})}}{P_{\mathfrak{m}}} \longrightarrow\!\!\!\!\rightarrow \{\pm 1\} = \mathrm{Gal}\left(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}\right) \\ \wr\!\| \\ (\mathbb{Z}/p)^* \end{array}$$

There are only two homomorphisms $(\mathbb{Z}/p)^* \to \{\pm 1\}$,

the trivial one and $n \longmapsto \left(\frac{n}{p}\right)$. Since the

Artin map is surjective it is nontrivial, so we get

$$\left(\frac{p^*}{q}\right) = \left(\frac{q}{p}\right).$$